Brief Announcement: Lower Bounds for Asymptotic Consensus in Dynamic Networks[*]

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In the *asymptotic consensus* problem a set of agents, each starting from an initial value in $\mathbb{R}^d$, update their values such that all agents' values converge to a common value within the convex hull of initial values. The problem is closely related to the *approximate consensus* problem, in which agents have to irrevocably decide on values that lie within a predefined distance $\varepsilon > 0$ of each other. The latter is weaker than the *exact consensus* problem in which agents need to decide on the same value. Both the asymptotic and the approximate consensus problems have not only a variety of applications in the design of man-made control systems like sensor fusion [1], clock synchronization [8], formation control [6], rendezvous in space [9], or load balancing [5], but also for analyzing natural systems like flocking [11], firefly synchronization [10], or opinion dynamics [7]. These problems often have to be solved under harsh-environmental restrictions: with limited computational power and local storage, under restricted communication abilities, and in presence of communication uncertainty.

In this work we study asymptotic consensus in round-based computational models with a dynamic communication topology whose directed communication graphs are chosen each round from a predefined set of communication graphs, the so-called *network model*. In previous work [2], Charron-Bost et al. showed that asymptotic consensus is solvable precisely within *rooted network models* in which all communication graphs contain rooted spanning trees. These rooted spanning trees need not have any edges in common and can change from one round to the next.

An interesting special case of rooted network models are network models whose graphs are *non-split*, that is, any two agents have a common incoming neighbor. Their prominent role is motivated by two properties: (i) They occur as communication graphs in benign classical distributed failure models. For example, in synchronous systems with crashes, in asynchronous systems with a minority of crashes, and synchronous systems with send omissions. (ii) In [2], Charron-Bost et al. showed that non-split graphs also play a central role in arbitrary rooted network models: they showed that any product of $n - 1$ rooted graphs with n nodes is non-split, allowing to transform asymptotic consensus algorithms for non-split network models into their *amortized* variants for rooted models.

Interestingly, solvability in any rooted network model is already provided by deceptively simple algorithms [2]: so-called *averaging* or *convex combination* algorithms, in which agents repeatedly broadcast their current value, and update it to some weighted average of the values they received in this round. One instance, proposed by Charron-Bost et al. [3] is the *midpoint algorithm*, in which agents update their value to the midpoint of the set of received values, i.e., the average of the smallest and the largest of the received values.

Regarding time complexity, for dimension $d = 1$, the midpoint algorithm was shown to

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have a contraction rate of $\frac{1}{2}$ in non-split network models [3]. This is *optimal* for "memoryless" averaging algorithms, which only depend on the values received in the current round [3].

The question arises whether non-averaging or non-memoryless algorithms, i.e., algorithms that (i) do not necessarily set their output values to within the convex hull of previously received values or (ii) whose output is a function not only of the previously received values, allow faster contraction rates. Indeed, algorithms that violate (i) and (ii) are studied in the literature. As an example for (i), consider the algorithm where each agent sends an equal fraction of its current output value to all out-neighbors and sets its output to the sum of values received in the current round. Note that the algorithm is not a convex combination algorithm as its output may lie outside the convex hull of the values of its in-neighbors. However, it solves asymptotic consensus algorithm for a fixed directed communication graph. Other examples of algorithms that violate (i) and (ii) are from control theory, where the usage of overshooting fast second-order controllers is common.

Contribution

In this work, we prove lower bounds on the contraction rate of any asymptotic consensus algorithm. All lower bounds hold regardless of the structure of the algorithm. In particular, algorithms can be full-information and agents can set their outputs outside the convex hull of received values. This, e.g., includes using higher-order filters in contrast to the 0-order filters of averaging algorithms.

The proof strategy is as follows: The central idea is the concept of the *valency of a configuration* of an asymptotic consensus algorithm, defined as the set of limits reachable from this configuration. By studying the changes in valency along executions, we infer bounds on the contraction rate. Notably, the lower bounds are valid for arbitrary dimensions.

Note that if exact consensus is solvable in network model $\mathcal{N}$, an optimal contraction rate of 0 can be achieved. Otherwise, we show the following non trivial bounds:

- We show a tight lower bound of $1/3$ in non-split network models with $n = 2$ agents.

- We prove that the contraction rate is lower bounded by $1/2$ in a system with $n \geq 3$ agents and $\mathrm{deaf}(G) \subseteq \mathcal{N}$ where, for an arbitrary communication graph G, $\mathrm{deaf}(G) = \{F_1, \dots, F_n\}$ and F_i is derived from G by making agent i *deaf* in F_i, i.e., removing the incoming links of i in G. Additionally we show tightness for $d \in \{1, 2\}$ dimensional values.

- The study of the valencies' topological structure with respect to the network model where the asymptotic consensus algorithm is executed in, reveals that any asymptotic consensus algorithm must have a contraction rate of at least $1/(D + 1)$, where D is the so-called α-diameter of $\mathcal{N}$. This generalizes the previous two lower bounds.

Tightness for $1/3$ and $1/2$ results from the combination with algorithms presented in [3] and [4]. Together with the algorithm for arbitrary dimensions d with contraction rate $\frac{d}{d+1}$ in non-split models [4] the bounds in Table 1 follow.

Furthermore we extend our results on contraction rates to derive new lower bounds on the decision time of any approximate consensus algorithms in non-split network models: $\log_3 \frac{\Delta}{\varepsilon}$ for $n = 2$, and in case $n \geq 3$, $\log_2 \frac{\Delta}{\varepsilon}$ for models with $\mathrm{deaf}(G)$ for some communication graph G, and $\log_{D+1} \frac{\Delta}{\varepsilon n}$ for arbitrary models in which exact consensus is not solvable. Again, deciding versions of the two asymptotic consensus algorithms for $n = 2$ and $n \geq 3$ from [3], have matching time complexity in non-split network models that include some $\mathrm{deaf}(G)$; showing optimality of these algorithms also for solving approximate consensus.

agents	dimension	network model in which exact consensus is unsolvable		
		non-split with deaf graphs	$\subseteq$ non-split $\subseteq$	rooted
$n = 2$	$d \geq 1$	$\frac{1}{3}$ *	$\frac{1}{3}$ *	$\frac{1}{3}$ *
$n \geq 3$	$d \in \{1,2\}$	$\frac{1}{2}$ *	$\left[\frac{1}{D+1}\,^*, \frac{1}{2}\right]$	$\left[\frac{1}{D+1}\,^*, \sqrt[n-1]{\frac{1}{2}}\right]$
	$d \geq 3$	$\left[\frac{1}{2}\,^*, \frac{d}{d+1}\right]$	$\left[\frac{1}{D+1}\,^*, \frac{d}{d+1}\right]$	$\left[\frac{1}{D+1}\,^*, \sqrt[n-1]{\frac{d}{d+1}}\right]$

Table 1 Summary of lower and upper bounds on contraction rates if consensus is not solvable. New lower bounds proved in this work are marked with a *. The three right columns distinguish between the case the network model is (i) non-split and contains deaf(G) for some communication graph G, (ii) is non-split, and (iii) is rooted.

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