Rewriting Techniques: TD 7

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Two terms s and t are **joinable** (written $s \downarrow t$) w.r.t. a reduction $\rightarrow$ if there is k such that $s \rightarrow^* k \leftarrow^* t$.

A reduction $\rightarrow$ is called:

- **terminating** if there is no infinite descending chain $a_0 \rightarrow a_1 \rightarrow \dots$;
- **locally confluent** if $v_1 \leftarrow u \rightarrow v_2$ implies $v_1 \downarrow v_2$;
- **confluent** if $v_1 \leftarrow^* u \rightarrow^* v_2$ implies $v_1 \downarrow v_2$;

A pair $(A, \leq)$, where $\leq$ is a binary relation on the set A, is a **well quasi-ordering** (wqo) if $\leq$:

- is a *quasi-order*, i.e. $\leq$ is reflexive and transitive;
- is *well-founded*, i.e. there are no infinite strictly decreasing sequences $a_0 > a_1 > a_2 > \dots$ in A;
- does not have *infinite anti-chains*, i.e. it does not exists an infinite subset I of A such that for each $a, b \in I$, $a \not\leq b$ and $b \not\leq a$.

Exercise 1 :
Which of the following are true? Give a justification or a counter-example.

1. Every locally confluent TRS is confluent.
2. Every confluent TRS is terminating.
3. If $\mathcal{R}$ is a non-terminating TRS then there are terms u, v such that $u \rightarrow^*_{\mathcal{R}} v$ and $u \trianglelefteq v$ (where $\trianglelefteq$ is the sub-term relation).
4. If $\mathcal{R}$ is terminating TRS, then $\rightarrow^*_{\mathcal{R}}$ is a wqo on $T(\mathcal{F})$.

Solution:

(1) False. Consider for instance $\mathcal{R}_2$ from the solution of Exercise 4.3 of TD5:

$$\mathtt{a} \rightarrow \mathtt{b},\ \mathtt{b} \rightarrow \mathtt{a},\ \mathtt{a} \rightarrow \mathtt{0},\ \mathtt{b} \rightarrow \mathtt{1}$$

represented by the following diagram.

$$\begin{tikzcd} \mathtt{0} & \mathtt{a} \arrow[l] \arrow[r, bend left] & \mathtt{b} \arrow[l, bend left] \arrow[r] & \mathtt{1} \end{tikzcd}$$

(2) False. Consider the TRS $\{a \rightarrow a\}$.

(3) True. Let $u_1 \rightarrow_{\mathcal{R}} u_2 \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} u_n \rightarrow_{\mathcal{R}} \dots$ be an infinite sequence of rewriting. We may assume that all u_i are ground (replace all variables with a constant). Since $\trianglelefteq$ is a wqo on $T(\mathcal{F})$ for finite $\mathcal{F}$, there are no infinite anti-chains and therefore $u_i \trianglelefteq u_j$ holds for two indices $i < j$.

(4) False. Consider $\mathcal{R}$ empty. A sequence $\mathtt{a}, \mathtt{f(a)}, \mathtt{f(f(a))}, \dots$ of distinct terms is an infinite anti-chain.

Exercise 2 :
Is the following rewrite system confluent and terminating?

$$\begin{aligned}
\texttt{sel}(\texttt{0}, \texttt{cons}(x, l)) &\to x \\
\texttt{sel}(\texttt{succ}(n), \texttt{cons}(x, l)) &\to \texttt{sel}(n, l) \\
\texttt{from}(n) &\to \texttt{cons}(n, \texttt{from}(\texttt{succ}(n))) \\
\texttt{first}(\texttt{0}, l) &\to \texttt{nil} \\
\texttt{first}(\texttt{succ}(n), \texttt{cons}(x, l)) &\to \texttt{cons}(x, \texttt{first}(n, l))
\end{aligned}$$

Solution:

It is confluent as it is orthogonal (left-linear and has no non-trivial critical pairs). It does not terminate (see third rule).

Exercise 3 :
Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two confluent TRS such that $\leftarrow_{\mathcal{R}_2} \circ \rightarrow_{\mathcal{R}_1} \subseteq \rightarrow_{\mathcal{R}_1} \circ \leftarrow_{\mathcal{R}_2}$.

1. Prove that $\leftarrow^*_{\mathcal{R}_1} \circ \rightarrow^*_{\mathcal{R}_1 \cup \mathcal{R}_2} \subseteq \rightarrow^*_{\mathcal{R}_1 \cup \mathcal{R}_2} \circ \leftarrow^*_{\mathcal{R}_1 \cup \mathcal{R}_2}$
2. Show that $\mathcal{R}_1 \cup \mathcal{R}_2$ is confluent.

Solution:

(1) Assume $u \leftarrow^*_{\mathcal{R}_1} t \rightarrow^n_{\mathcal{R}_1 \cup \mathcal{R}_2} v$. We prove the result by induction on n. If $n = 0$, this is trivial: $t = v \rightarrow^*_{\mathcal{R}_1} u$. Otherwise, there are two cases.

- the first step is a reduction by $\mathcal{R}_1$:

  $$u \leftarrow^*_{\mathcal{R}_1} t \rightarrow_{\mathcal{R}_1} t' \rightarrow^{n-1}_{\mathcal{R}_1 \cup \mathcal{R}_2} v$$

  By confluence of $\mathcal{R}_1$, $u \rightarrow^*_{\mathcal{R}_1} u' \leftarrow^*_{\mathcal{R}_1} t' \rightarrow^{n-1}_{\mathcal{R}_1 \cup \mathcal{R}_2} v$ and it is sufficient to apply the induction hypothesis.

- the first step is a reduction by $\mathcal{R}_2$:

  $$u \leftarrow^k_{\mathcal{R}_1} t \rightarrow_{\mathcal{R}_2} t' \rightarrow^{n-1}_{\mathcal{R}_1 \cup \mathcal{R}_2} v$$

  Then, by induction on k, thanks to the inclusion $\leftarrow_{\mathcal{R}_2} \circ \rightarrow_{\mathcal{R}_1} \subseteq \rightarrow_{\mathcal{R}_1} \circ \leftarrow_{\mathcal{R}_2}$, it holds $u \rightarrow_{\mathcal{R}_2} \circ \leftarrow^k_{\mathcal{R}_1} t'$. We may again apply the induction hypothesis.

(2) From the previous question, by symmetry $\leftarrow_{\mathcal{R}_1 \cup \mathcal{R}_2} \circ \rightarrow^*_{\mathcal{R}_1 \cup \mathcal{R}_2} \subseteq \rightarrow^*_{\mathcal{R}_1 \cup \mathcal{R}_2} \circ \leftarrow^*_{\mathcal{R}_1 \cup \mathcal{R}_2}$. Hence $\mathcal{R}_1 \cup \mathcal{R}_2$ is semi-confluent and therefore confluent.

Let $\texttt{status}(f) \in \{\text{mul}, \text{lex}\}$ (i.e. multiset order or lexicographic order) a *status* function on Σ and let $>$ be a strict order on Σ. The **recursive path order** $>_{\text{rpo}}$ on $T(\Sigma, V)$ induced by $>$ is defined as follows. $s >_{\text{rpo}} t$ if and only if one of the following holds:

1. t is a variable appearing in s and $s \neq t$, or

let $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$,

2. there exists $i \in [1, m]$ such that $s_i \geq_{\text{rpo}} t$, or
3. $f > g$ and $s >_{\text{rpo}} t_j$ for all $j \in [1, n]$, or
4. $f = g$, for all $j \in [1, n]$ it holds $s >_{\text{rpo}} t_j$ and $(s_1, \ldots, s_m)(>_{\text{rpo}})_{\texttt{status}(f)}(t_1, \ldots, t_m)$.

A **polynomial interpretation on integers** is the following:

- a subset A of $\mathbb{N}$;
- for every symbol f of arity n, a polynomial $\mathtt{P}_f \in \mathbb{N}[X_1, \ldots, X_n]$;
- for every $a_1, \ldots, a_n \in A$, $\mathtt{P}_f(a_1, \ldots, a_n) \in A$;
- for every $a_1, \ldots, a_i > a_i', \ldots, a_n \in A$, $\mathtt{P}_f(a_1, \ldots, a_i, \ldots, a_n) > \mathtt{P}_f(a_1, \ldots, a_i', \ldots, a_n)$;

Then $(A, (\mathtt{P}_f)_f, >)$ is a well-founded monotone algebra.

Exercise 4 :
Let $\mathcal{R}$ be the following TRS:

$$\begin{aligned} \mathtt{f}(\mathtt{f}(x,y),z) &\to \mathtt{f}(x,\mathtt{f}(y,z)) \\ \mathtt{f}(y,\mathtt{f}(x,z)) &\to \mathtt{f}(x,x) \end{aligned}$$

1. Show that the termination of $\mathcal{R}$ cannot be proved with RPO.
2. Show that $\mathcal{R}$ terminates by defining a suitable polynomial interpretation over integers.

Solution:

(1) If we consider $\mathtt{status}(\mathtt{f}) = \text{lex}$ then $y >_{\text{rpo}} x$ (from the second rule) does not holds, whereas $\{\!|\mathtt{f}(x,y), z|\!\}(>_{\text{rpo}})_{\text{mul}}\{\!|x, \mathtt{f}(y,z)|\!\}$ does not holds when we consider $\mathtt{status}(\mathtt{f}) = \text{mul}$.

(2) We consider $A = \mathbb{N} \setminus \{0,1,2\}$ and $P_{\mathtt{f}} = X^2 + XY$. It holds that

$$P_{\mathtt{f}(\mathtt{f}(x,y),z)} = X^4 + 2X^3Y + X^2y^2 + X^2Z + XYZ > X^2 + Y^2X + XYZ = P_{\mathtt{f}(x,\mathtt{f}(y,z))}$$
$$P_{\mathtt{f}(y,\mathtt{f}(x,z))} = Y^2 + X^2Y + XZ > 2X^2 = P_{\mathtt{f}(x,x)}$$

Exercise 5 :
An order $>$ is total if, for all x, y, $x < y$, $x = y$ or $x > y$. Prove that if $>$ is a total order on function symbols and $>_{lpo}$ is an order, then $>_{lpo}$ is total on closed terms.

Solution:

By induction on the size of terms (i.e. suppose that the property holds for terms of size less or equal than n, show that it holds for terms of size less or equal than $n+1$). The base case for constants symbol is trival as $>$ is a total order. For the inductive case, simply apply the definition of LPO.

A **completion procedure** is a program that accepts as input a finite set of identities E_0 and a reduction order $>$, and generate a (finite or infinite) sequence (called run) $(E_0, R_0), (E_1, R_1), (E_2, R_2), \ldots$ where $R_0 = \emptyset$, by applying the rules:

$$\frac{E, R \quad s \leftarrow_R u \rightarrow_R t}{E \cup \{s = t\}, R}\text{Deduce} \qquad \frac{E \cup \{s \doteq t\}, R \quad s > t}{E, R \cup \{s \to t\}}\text{Orient}$$

$$\frac{E \cup \{s = s\}, R}{E, R}\text{Delete} \qquad \frac{E \cup \{s \doteq t\}, R \quad s \to_R u}{E \cup \{u = t\}, R}\text{Simplify}$$

$$\frac{E, R \cup \{s \to t\} \quad t \to_R u}{E, R \cup \{s \to u\}}\text{Compose} \qquad \frac{E, R \cup \{s \to t\} \quad s \overset{\sqsupset}{\to}_R u}{E \cup \{u = t\}, R}\text{Collapse}$$

where $s \doteq t$ if and only if $s = t$ or $t = s$, whereas $s \overset{\sqsupset}{\to}_R u$ is used to express that s is reduced to u by a rule $l \to r$ of R such that each sub-term of l is not an instance of s. A special case of *Deduce* is to apply it only if (s,t) is a critical pair. Most completion procedures use the rule *Deduce* only in this way. The goal of these procedures is to transform an initial pair $(E_0, \emptyset)$ into a pair $(\emptyset, \mathcal{R})$ such that $\mathcal{R}$ is a **convergent** TRS such that for every equivalence $s = t \in E_0$, s and t are joinable by $\mathcal{R}$.

Exercise 6 :
Consider the single equation $\mathtt{I}(x) \times (x \times y) = y$. Compute a convergent TRS for the equational theory defined by this equation. *hint: you don't need to use all the rules.*

Solution:
For this Exercise, we will just use the rules *Orient* and *Deduce*, as follows:

1. Update R by using *Orient* on an element of E,
2. Check for a critical pair (s, t) in R s.t. s are t are not joinable w.r.t. R.
   - if (s, t) exists, then R is not locally confluent. *Deduce* $s = t$ (which is added to E).
   - if (s, t) does not exists, R is locally confluent.
3. if E is empty, terminate. Otherwise, go to (1).

Let $E_0 = \{\mathtt{I}(x) \times (x \times y) = y\}$ and $R_0 = \emptyset$. By applying the *Orient* rule we obtain $E_1 = \emptyset$ and $R_1 = \{\mathtt{I}(x) \times (x \times y) \to y\}$. The only critical peak in R_1 is $\mathtt{I}(\mathtt{I}(x)) \times (\mathtt{I}(x) \times (x \times y))$ with critical pair $(\mathtt{I}(\mathtt{I}(x)) \times y, x \times y)$. R_1 is not locally confluent. By applying *Deduce* we obtain $E_2 = \{\mathtt{I}(\mathtt{I}(x)) \times y = x \times y\}$ and $R_2 = R_1$. By orienting the new identity we obtain $E_3 = \emptyset$ and

$$R_3 = \{\mathtt{I}(x) \times (x \times y) \to y,\ \mathtt{I}(\mathtt{I}(x)) \times y \to x \times y\}$$

which has a new critical peak $\mathtt{I}(\mathtt{I}(x)) \times (\mathtt{I}(x) \times y)$. R_3 is not locally confluent. Therefore, we apply *Deduce* to the critical pair $(x \times (\mathtt{I}(x) \times y, y)$ of this new critical peak and obtain $E_4 = \{x \times (\mathtt{I}(x) \times y = y\}$ and $R_4 = R_3$.
We now apply *Orient* and obtain $E_5 = \emptyset$ and

$$R_5 = \{\mathtt{I}(x) \times (x \times y) \to y,\ \mathtt{I}(\mathtt{I}(x)) \times y \to x \times y,\ x \times (\mathtt{I}(x) \times y \to y\}$$

It's easy to prove that R_5 is a convergent TRS. Termination is trivial, whereas confluence follows by Newman's Lemma from its local confluency, which holds since all of its critical pairs are joinable. For example, consider the critical peak $x \times (\mathtt{I}(x) \times (\mathtt{I}(\mathtt{I}(x)) \times y))$ which is associated with the following diagram

$$\begin{tikzcd}[column sep=small] & x \times (\mathtt{I}(x) \times (\mathtt{I}(\mathtt{I}(x)) \times y)) \arrow[dl] \arrow[dr] & \\ \mathtt{I}(\mathtt{I}(x)) \times y \arrow[rr] & & x \times y \end{tikzcd}$$

Exercise 7 :
Let R and B be two well-founded relations such that

$$RB \subseteq R \cup (B(R \cup B)^*)$$

We say that (x, y) is a blue (resp. red) edge if $(x, y) \in B$ (resp. $(x, y) \in R$). A blue path is a sequence of blue edges. A node x is infinite if there is an infinite $R \cup B$-sequence starting from x. Otherwise x is said to be finite. An infinite node x is red if, from x, every blue edge arrives at a finite node.

1. Prove that, from every infinite node, there is a blue path arriving at a red node.
2. Prove that $RB^* \subseteq R \cup (B(R \cup B)^*)$.
3. Prove that, from every red node, there is a red edge arriving at a red node.
4. Conclude that $R \cup B$ is well-founded.
5. As a consequence, prove that, in particular, $R \cup B$ is well-founded if R and B are well-founded and $R \cup B$ is transitive.