Rewriting Techniques: TD 6

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A **multi-context** C is a term with distinguished variables $\square_1, \ldots, \square_n$ occurring exactly once. Replacing them by terms $t_1, \ldots, t_n$ respectively is denoted by $C[t_1, \ldots, t_n]$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two disjoint signatures. A symbol is of **color** $k \in \{1, 2\}$ if it belongs to $\mathcal{F}_k$. A term t is of color k if it's not a variable and every symbol in it is of color k. We denote with $\overline{k}$ the other possible color of k, i.e. $3 - k$.

Let t be a term with symbols in $\mathcal{F}_1 \cup \mathcal{F}_2$. We define $\text{cap}(t)$ and $\text{aliens}(t)$ respectively as

$$\text{cap}(t) = \begin{cases} x & \text{if } t = x \text{ is a variable} \\ C & \text{if } t = C[t_1, \ldots, t_n] \text{ where } C \text{ is of color } k \in \{1,2\} \text{ and} \\ & t_1, \ldots, t_n \text{ are headed by symbols of color } \overline{k} \end{cases}$$

$$\text{aliens}(t) = \begin{cases} \emptyset & \text{if } t \text{ is a variable} \\ \{\!|t_1, \ldots, t_n|\!\} & \text{if } t = C[t_1, \ldots, t_n] \text{ where } C \text{ is of color } k \in \{1,2\} \text{ and} \\ & t_1, \ldots, t_n \text{ are headed by symbols of color } \overline{k} \end{cases}$$

The **rank** of a term t, denoted with $\text{rk}(t)$, is the maximum number of color layers in t, i.e. $\text{rk}(t) = 1 + \max_{a \in \text{aliens}(t)}(\text{rk}(a))$.

Exercise 1 :
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two disjoint signatures and let $\mathcal{R}_1$, $\mathcal{R}_2$ be two TRSs on $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively such that $\rightarrow_{\mathcal{R}_1}$ terminates on $T_1 = T(\mathcal{F}_1, V)$ and $\rightarrow_{\mathcal{R}_2}$ terminates on $T_2 = T(\mathcal{F}_2, V)$, where V is a set of variables. Let $\rightarrow$ be the rewrite relation on $T = T(\mathcal{F}_1 \cup \mathcal{F}_2, V)$ generated by $\mathcal{R}_1 \cup \mathcal{R}_2$.

1. Prove that for each term $t, u \in T$, if $t \rightarrow u$ then $\text{rk}(t) \geq rk(u)$.
2. Prove that if $\mathcal{R}_1$ and $\mathcal{R}_2$ do not have any rules of the form $l \rightarrow x$ where x is a variable, then $\rightarrow$ terminates.
3. Prove that $\mathcal{R}_1 = \{\mathtt{a}(\mathtt{0}, \mathtt{1}, x) \rightarrow \mathtt{a}(x, x, x)\}$ and $\mathcal{R}_2 = \{\mathtt{m}(x, y) \rightarrow x,\ \mathtt{m}(x, y) \rightarrow y\}$ are terminating, whereas $\mathcal{R}_1 \cup \mathcal{R}_2$ is not.

Solution:

(1) The proof is by induction on $\text{rk}(t)$. We need to distinguish two cases:

- The reduction that leads to u is in $\text{cap}(t)$. If u is a variable, then $\text{rk}(u) = 1 \leq \text{rk}(t)$. If $\text{cap}(t)$ and $\text{cap}(u)$ have distinct colors, then u is an alien of t and $\text{rk}(u) < \text{rk}(t)$. Otherwise it must holds that $\text{cap}(t) \rightarrow_{\mathcal{R}_1} \text{cap}(u)$ or $\text{cap}(t) \rightarrow_{\mathcal{R}_2} \text{cap}(u)$ (if $\text{cap}(t)$ is of color 1 or 2 respectively) and $\text{aliens}(u) \subseteq \text{aliens}(t)$. Therefore, $\text{rk}(u) \leq \text{rk}(t)$.
- Let $t = C[t_1, \ldots, t_n]$ where $C = \text{cap}(t)$. Suppose now that reduction is in $t_i \in \text{aliens}(t)$, $i \in [1, n]$, and it reduces t_i to t'_i (therefore, $u = C[t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_n]$). By induction hypothesis (since $\text{rk}(t_i) < \text{rk}(t)$), $\text{rk}(t'_i) \leq \text{rk}(t_i)$. By definition of rk it holds that $\text{rk}(t) = \text{rk}(C[t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n]) \geq \text{rk}(C[t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_n]) = \text{rk}(u)$.

(2) Let t be a term. The proof is by well-founded induction on $(\text{rk}(t), \text{cap}(t), \text{aliens}(t))$ w.r.t. the well-founded order $(>, (\rightarrow^+_{\mathcal{R}_1} \cup \rightarrow^+_{\mathcal{R}_2}), \rightarrow_{\text{mul}})_{\text{lex}}$. Notice that, since $\rightarrow_{\mathcal{R}_1}$ and $\rightarrow_{\mathcal{R}_2}$ are terminating, the relation $(\rightarrow^+_{\mathcal{R}_1} \cup \rightarrow^+_{\mathcal{R}_2}) \subseteq (T(\mathcal{F}_1, V) \cup T(\mathcal{F}_2, V))^2$ that maps elements of $T(\mathcal{F}_1, V)$ to elements of $T(\mathcal{F}_1, V)$ and elements of $T(\mathcal{F}_2, V)$ to elements of $T(\mathcal{F}_2, V)$ is well-founded. $\rightarrow_{\text{mul}}$ is also well-founded as explained in the last point of the proof (below). If t is irreducible, then $\rightarrow$ terminates on it. Instead, if $t \rightarrow u$, then we need to consider the following three cases:

Require: A finite set E of identities and a reduction order $>$

Ensure: A finite convergent (terminating and confluent) rewrite system R equivalent to E if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully

$R_0 := \varnothing$; $E_0 := E$; $i := 0$; all identities of E are unmarked
while $E_i \neq \varnothing$ or there is an unmarked rule in R_i **do**
  while $E_i \neq \varnothing$ **do**
    Choose an identity $(s,t) \in E$ and reduce s and t to some R_i-normal forms $\tilde{s}$ and $\tilde{t}$
    if $\tilde{s} = \tilde{t}$ **then**
      $R_{i+1} := R_i$; $E_{i+1} := E_i \setminus \{(s,t)\}$; $i := i+1$
    else if $\tilde{s} \not> \tilde{t} \wedge \tilde{t} \not> \tilde{s}$ **then**
      terminates with output FAIL
    else
      let l and r such that $\{l,r\} = \{\tilde{s},\tilde{t}\}$ and $l > r$
      $R_{i+1} := \{(g,\tilde{d}) \mid (g,d) \in R_i \wedge g$ cannot be reduced with $l \to r \wedge \tilde{d}$ is a $R_i \cup \{(l,r)\}$-normal form of $d\} \cup \{(l,r)\}$
      (l,r) is not marked and $(g,\tilde{d})$ is marked in R_{i+1} iff (g,d) is marked in R_i
      $E_{i+1} := (E_i \setminus \{(s,t)\}) \cup \{(g',d) \mid (g,d) \in R_i \wedge g$ can be reduced to g' with $l \to r\}$
      $i := i+1$
    end if
  end while
  if there is an unmarked rule in R_i **then**
    let (l,r) be such a rule
    $R_{i+1} := R_i$
    $E_{i+1} := \{(s,t) \mid (s,t)$ is a critical pair of (l,r) with itself or with a marked rule in $R_i\}$
    Mark (l,r) ; $i := i+1$
  end if
end while
return R_i

Figure 1: Huet's completion procedure

- $\mathrm{rk}(t) = 1$, i.e. t is of color $k \in \{1,2\}$. Then u is of the same color and therefore $t \to_{\mathcal{R}_k} u$. We can apply the induction hypothesis (u terminates) and conclude that also t terminates.
- $\mathrm{rk}(t) > 1$ and the reduction is in $C = \mathrm{cap}(t)$. Let k be the color of C. Since the rules are non-collapsing, then $\mathrm{cap}(u)$ is of color k and, by well-foundness of $\to_{\mathcal{R}_k}$ we can apply the induction hypothesis (u terminates) and conclude that, t also terminates.
- Lastly, $\mathrm{rk}(t) > 1$ and the reduction is in some alien $a \in \mathrm{aliens}(t)$, that is reduced to a'. Then, since $\mathrm{rk}(a) < \mathrm{rk}(t)$, we can apply the induction hypothesis and conclude that a is terminating. Since the rules are non-collapsing, it holds that $\mathrm{cap}(t) = \mathrm{cap}(u)$ and, from $\mathrm{aliens}(u) = \mathrm{aliens}(t) - \{\!|a|\!\} + \{\!|a'|\!\}$, all aliens of t and u are terminating. Therefore, it holds that $\to_{\mathrm{mul}}$ is a well-founded strict order when restricted to elements of $\mathrm{aliens}(t)$. Since $\mathrm{aliens}(t) \to_{\mathrm{mul}} \mathrm{aliens}(u)$, u terminates and so does t.

(3) For $\mathcal{R}_1$ consider the order $>$ where $t > s$ if and only if

$$\{\!|\ |u| \mid \exists p \in \mathrm{Pos}(t)\ t|_p = f(\mathtt{0},\mathtt{1},u)|\!\} >_{\mathrm{mul}} \{\!|\ |u| \mid \exists p \in \mathrm{Pos}(s)\ s|_p = f(\mathtt{0},\mathtt{1},u)|\!\}$$

where $|u|$ is the height of u. It's easy to show that $>$ is a simplification order such that for each terms $t \to s$ it holds $t > s$. Instead, the termination of $\mathcal{R}_2$ is trivial (consider the size of the term). Lastly, $\mathcal{R}_1 \cup \mathcal{R}_2$ does not terminate:

$$\mathtt{a}(\mathtt{m}(\mathtt{0},\mathtt{1}),\mathtt{m}(\mathtt{0},\mathtt{1}),\mathtt{m}(\mathtt{0},\mathtt{1})) \to \mathtt{a}(\mathtt{0},\mathtt{m}(\mathtt{0},\mathtt{1}),\mathtt{m}(\mathtt{0},\mathtt{1})) \to \mathtt{a}(\mathtt{0},\mathtt{1},\mathtt{m}(\mathtt{0},\mathtt{1})) \to \mathtt{a}(\mathtt{m}(\mathtt{0},\mathtt{1}),\mathtt{m}(\mathtt{0},\mathtt{1}),\mathtt{m}(\mathtt{0},\mathtt{1}))$$

Exercise 2 :

1. Prove that the set of identities

$$\begin{aligned}
&(@(\mathtt{nil}, x),\ x),\\
&(@(\mathtt{cons}(x, y), z),\ \mathtt{cons}(x, @(y, z))),\\
&(\mathtt{rev}(\mathtt{nil}),\ \mathtt{nil}),\\
&(\mathtt{rev}(\mathtt{cons}(x, y)),\ @(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil})))
\end{aligned}$$

   can be oriented to give a convergent TRS. Let R this TRS.
2. Why the associativity A of @, $@(@(x, y), z) = @(x, @(y, z))$ is not a consequence of R?
3. Prove that you can complete (A, R). You can use Huet's completion procedure (Figure 1).
4. Show that Huet's completion fails to complete $(\{\mathtt{rev}(x) = @(x, x)\}, R)$.

Solution:

(1) We use LPO w.r.t. the order $\mathtt{rev} > @ > \mathtt{cons} > \mathtt{nil}$ to orient the rules as follows:

$$\begin{aligned}
@(\mathtt{nil}, x) &>_{lpo} x\\
@(\mathtt{cons}(x, y), z) &>_{lpo} \mathtt{cons}(x, @(y, z))\\
\mathtt{rev}(\mathtt{nil}) &>_{lpo} \mathtt{nil}\\
\mathtt{rev}(\mathtt{cons}(x, y)) &>_{lpo} @(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil}))
\end{aligned}$$

$R = \{@(\mathtt{nil}, x) \to x,\ @(\mathtt{cons}(x, y), z) \to \mathtt{cons}(x, @(y, z)),\ \mathtt{rev}(\mathtt{nil}) \to \mathtt{nil},\ \mathtt{rev}(\mathtt{cons}(x, y)) \to @(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil}))\}$ is a TRS with no critical pairs. Therefore, by critical pairs Lemma, R is locally confluent. Moreover, since we can use LPO to prove its termination, by Newman's Lemma (which implies confluency of R), R is convergent.

Notice that if we change the orientation of the last rule to

$$@(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil})) >_{lpo} \mathtt{rev}(\mathtt{cons}(x, y))$$

we obtain a TRS with a critical pair derived from the following diagram:

$$\begin{tikzcd}
& @(\mathtt{rev}(\mathtt{nil}), \mathtt{cons}(x, \mathtt{nil})) \arrow[dl] \arrow[dr] & \\
@(\mathtt{nil}, \mathtt{cons}(x, \mathtt{nil})) \arrow[d] & & \mathtt{rev}(\mathtt{cons}(x, \mathtt{nil})) \\
\mathtt{cons}(x, \mathtt{nil}) & &
\end{tikzcd}$$

Which is not a convergent critical pair. Therefore this orientation is not enough to get a convergent TRS and we would need to apply a completion procedure.

(2) $@(@(x, y), z)$ and $@(x, @(y, z))$ are already in normal form w.r.t. R and therefore associativity is not a consequence of R.

(3) Consider R_0 as the LPO of (1), equal to

$$\begin{aligned}
@(\mathtt{nil}, x) &\to\ x,\\
@(\mathtt{cons}(x, y), z) &\to \mathtt{cons}(x, @(y, z)),\\
\mathtt{rev}(\mathtt{nil}) &\to \mathtt{nil},\\
\mathtt{rev}(\mathtt{cons}(x, y)) &\to @(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil}))
\end{aligned}$$

and $R_0 = \emptyset$. Let $E_0 = \{(@(@(x', y'), z'),\ @(x', @(y', z')))\} = A$ We now apply the procedure from Line 2 of the pseudocode. by Line 4, we take the identity $(@(@(x', y'), z'),\ @(x', @(y', z')))$ and reduce it to a normal form using R_0 (by the last point in the exercise, the identity is already in normal form). By definition of LPO, the test in Line 7 fails (fortunately) and we will execute from Line 10, with $l = (@(@(x', y'), z') > @(x', @(y', z'))) = r$. Line 11 simply add (l, r) to R_0, so that $R_1 = \{(l, r)\} \cup R_0$. Then, $E_1 = \emptyset$ by Line 13. The procedure will continue by evaluating the conditional at line 17. As already shown in (1), R_0 does not have any critical pairs. Then assume that the rules of R_0 are chosen before (l, r), resulting in them being now marked. We now only need to consider the critical pairs between the associativity rule and the other rules. We derive the following diagrams based on the substitutions $\sigma_{5,1} = [x'/\texttt{nil}]$, $\sigma_{5,2} = [x'/\texttt{cons}(x, y)]$ and $\sigma_{5,5} = [x'/@(x, y)]$, where $\sigma_{i,j}$ is the mgu that can be used to compute a critical pair between the rules i and j:

$$
\begin{tikzcd}
 & @(@(\texttt{nil}, y), z) \arrow[dl] \arrow[dr] & \\
@(\texttt{nil}, @(y, z)) \arrow[rr] & & @(y, z)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & @(@(\texttt{cons}(x, y), z), w) \arrow[dl] \arrow[dr] & \\
@(\texttt{cons}(x, y), @(z, w)) \arrow[dr] & & @(\texttt{cons}(x, @(y, z)), w) \arrow[dl] \\
 & \texttt{cons}(x, @(y, @(z, w))) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & @(@(@(x, y), z), w) \arrow[dl] \arrow[dr] & \\
@(@(x, y), @(z, w)) \arrow[ddr] & & @(@(x, @(y, z)), w) \arrow[d] \\
 & & @(x, @(@(y, z), w)) \arrow[d] \\
 & & @(x, @(y, @(z, w)))
\end{tikzcd}
$$

The critical pairs in these diagrams are

- $(@(\texttt{nil}, @(y, z)), @(y, z))$ where $@(\texttt{nil}, @(y, z)) >_{lpo} @(y, z)$,
- $(@(\texttt{cons}(x, y), @(z, w)), @(\texttt{cons}(x, @(y, z)), w))$ where

$$@(\texttt{cons}(x, y), @(z, w)) >_{lpo} @(\texttt{cons}(x, @(y, z)), w)$$

- $(@(@(x, y)@(z, w)), @(@(x, @(y, z)), w)$ where $@(@(x, @(y, z)), w) >_{lpo} @(@(x, y)@(z, w))$.

Notice how all three diagrams are confluent. For this reason, lets put aside the Huet's procedure and consider the TRS R_1:

$$
\begin{aligned}
@(\texttt{nil}, x) &\to x \\
@(\texttt{cons}(x, y), z) &\to \texttt{cons}(x, @(y, z)) \\
\texttt{rev}(\texttt{nil}) &\to \texttt{nil} \\
\texttt{rev}(\texttt{cons}(x, y)) &\to @(\texttt{rev}(y), \texttt{cons}(x, \texttt{nil})) \\
(@(@(x', y'), z') &\to @(x', @(y', z')))
\end{aligned}
$$

its easy to show that this TRS is terminating w.r.t. the LPO induced by the order the order $\texttt{rev} > @ > \texttt{cons} > \texttt{nil}$. Moreover, from the diagrams above, all critical pairs are convergent. As such, the TRS is locally confluent and by Newman's Lemma, is also convergent. Moreover, in this TRS it trivially holds that $(@(@(x', y'), z')$ and $@(x', @(y', z')))$ have the same normal form and the TRS is a completion for (A, R).

(4) For simplicity, let $R_0 = R$ and $E_0 = \{(\texttt{rev}(x), @(x, x))\}$ and suppose we start the evaluation of Huet's procedure from line 3. It holds that $\texttt{rev}(x) >_{lpo} @(x, x)$ and the two terms are already in R_0-normal form. The rules $(\texttt{rev}(\texttt{nil}), \texttt{nil})$ and $(\texttt{rev}(\texttt{cons}(x, y)), @(\texttt{rev}(y), \texttt{cons}(x, \texttt{nil})))$ can be reduced via $\texttt{rev}(x)$. As such, it will hold that

$$R_1 = \{(@(\texttt{nil}, x), x), (@(\texttt{cons}(x, y), z), \texttt{cons}(x, @(y, z))), (\texttt{rev}(x), @(x, x))\}$$

and

$$E_1 = \{(@(\texttt{nil}, \texttt{nil}), \texttt{nil}), (@(\texttt{cons}(x, y), \texttt{cons}(x, y)), @(\texttt{rev}(y), \texttt{cons}(x, \texttt{nil})))\}$$

The first identity of E_1 will be simply removed since the R_1-normal form of $@(\texttt{nil}, \texttt{nil})$ is exactly $\texttt{nil}$. Instead, the normal form of the second identity in E_1 is

$$(\texttt{cons}(x, @(y, \texttt{cons}(x, y))), @(@(y, y), \texttt{cons}(x, \texttt{nil})))$$

and is such that $@(@(y, y), \texttt{cons}(x, \texttt{nil})) >_{lpo} \texttt{cons}(x, @(y, \texttt{cons}(x, y)))$. Therefore, E and R will be updated so that $E_3 = \emptyset$ and

$$\begin{aligned} R_3 =\{&(@(\texttt{nil}, x), x), (@(\texttt{cons}(x, y), z), \texttt{cons}(x, @(y, z))), (\texttt{rev}(x), @(x, x)), \\ &(@(@(y, y), \texttt{cons}(x, \texttt{nil})), \texttt{cons}(x, @(y, \texttt{cons}(x, y))))\} \end{aligned}$$

The procedure will then search for critical pairs of R_3 and eventually find the critical pairs between $@(\texttt{cons}(x, y), z)$ and $@(@(y', y'), \texttt{cons}(x', \texttt{nil}))$, in particular w.r.t. the substitution $\sigma = [y'/\texttt{cons}(x, y), z/\texttt{cons}(x, y)]$ from which we derive the following diagram

$$\begin{array}{ccc} & @(@(\texttt{cons}(x, y), \texttt{cons}(x, y)), \texttt{cons}(x', \texttt{nil}))) & \\ \swarrow & & \searrow \\ @(\texttt{cons}(x, @(y, \texttt{cons}(x, y))), \texttt{cons}(x', \texttt{nil})) & & \texttt{cons}(x', @(\texttt{cons}(x, y), \texttt{cons}(x', \texttt{cons}(x, y)))) \\ \downarrow & & \\ \texttt{cons}(x, @(@(y, \texttt{cons}(x, y)), \texttt{cons}(x', \texttt{nil}))) & & \end{array}$$

the critical pair

$$(@(\texttt{cons}(x, @(y, \texttt{cons}(x, y))), \texttt{cons}(x', \texttt{nil})), \texttt{cons}(x', @(\texttt{cons}(x, y), \texttt{cons}(x', \texttt{cons}(x, y)))))$$

will be added to E and eventually selected in line 4. However, from the diagram we can see that the normal form of this critical pair is $(\tilde{s}, \tilde{t}) =$

$$(\texttt{cons}(x, @(@(y, \texttt{cons}(x, y)), \texttt{cons}(x', \texttt{nil}))), \texttt{cons}(x', @(\texttt{cons}(x, y), \texttt{cons}(x', \texttt{cons}(x, y)))))$$

and is such that $\tilde{s} \neq \tilde{t}$, $\tilde{s} \not>_{lpo} \tilde{t}$ and $\tilde{t} \not>_{lpo} \tilde{s}$ and therefore the completion procedure will return FAIL. The completion will also fail if we considered an order where $@(x, x) > \texttt{rev}(x)$, as implied by (1).

The following exercises are taken from last year exam.

Exercise 3 :
Is the following rewrite system confluent and terminating?

$$\begin{aligned}
\texttt{add}(\texttt{0}, y) &\to y \\
\texttt{add}(\texttt{s}(x), y) &\to \texttt{s}(\texttt{add}(x, y)) \\
\texttt{mul}(\texttt{0}, y) &\to \texttt{0} \\
\texttt{mul}(\texttt{s}(x), y) &\to \texttt{add}(y, \texttt{mul}(x, y)) \\
\texttt{add}(\texttt{add}(x, y), z) &\to \texttt{add}(x, \texttt{add}(y, z)) \\
\texttt{mul}(\texttt{add}(x, y), z) &\to \texttt{add}(\texttt{mul}(x, z), \texttt{mul}(y, z))
\end{aligned}$$

Exercise 4 :
Prove that a rewrite system $\mathcal{R}$ terminates by using a monotone polynomial interpretation on $D_k = \{n \in \mathbb{Z} \mid n > k\}$ $(k \in \mathbb{Z})$ if and only if it does so by using a polynomial interpretation on D_0.

Exercise 5 :
Let $\leq$ be a quasi-ordering on function symbols. Let $<=\leq - \leq^{-1}$ be its strict part and $\approx=\leq \cap \leq^{-1}$ be its associated equivalence relation. Prove that $<_{lpo}$ is transitive, stable (by substitution), monotone and contains the subterm relation.