Rewriting Techniques: TD 3

29-11-2018

Let Σ be a finite signature with one constant (i.e. there is $s \in \Sigma$ having arity 0). Let V be a set of variable and $T(\Sigma, V)$ the terms built from those sets. For $s \in \Sigma \cup V$ and a term $t \in T(\Sigma, V)$, we denote with $|t|_s$ the number of occurrences of s in t.

Let $>$ be a strict order on Σ. A function $w : \Sigma \cup V \to \mathbb{R}_0^+$ is called **admissible weight function** if and only if it satisfy the following properties:

1. There exists $w_0 \in \mathbb{R}_0^+ \setminus \{0\}$ s.t. $w(x) = w_0$ for all $x \in V$ and $w(c) \geq w_0$ for all constants $c \in \Sigma$.
2. If $f \in \Sigma$ is a unary function symbol of weight $w(f) = 0$ then f is the greatest element in Σ, i.e. $f \geq g$ for all $g \in \Sigma$.

An admissible weight function w is extended on every term $t \in T(\Sigma, V)$ as follows:

$$w(t) = \sum_{x \in V} w(x)|t|_x + \sum_{f \in \Sigma} w(f)|t|_f$$

Let $>$ be a strict order on Σ and $w : \Sigma \cup V \to \mathbb{R}_0^+$ be an admissible weight function. The **Knuth-Bendix order** (KBO) $>_{\text{kbo}}$ on $T(\Sigma, V)$ induced by $>$ and w is defined as follows: for $s, t \in T(\Sigma, V)$ we have $s >_{\text{kbo}} t$ if and only if $|s|_x \geq |t|_x$ for all $x \in V$ and $w(s) \geq w(t)$. Moreover, if $w(s) = w(t)$ then one of the following properties must hold:

1. There are a unary function f, $x \in V$ and $n \in \mathbb{N}^{\geq 1}$ s.t. $s = f^n(x)$ and $t = x$, or
2. there exist function symbols f, g s.t. $f > g$ and $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$, or
3. there exist a function symbol f such that $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$ and

$$(s_1, \ldots, s_m)(>_{\text{kbo}})_{\text{lex}}(t_1, \ldots, t_m).$$

Exercise 1 :
Using a KBO, prove the termination of:

1. $\{\mathtt{l}(x) + (y + z) \to x + (\mathtt{l}(\mathtt{l}(y)) + z),\ \mathtt{l}(x) + (y + (z + w)) \to x + (z + (y + w))\ \}$
2. $\{\ \mathtt{r}^n(\mathtt{l}^k(x)) \to \mathtt{l}^k(\mathtt{r}^m(x))\ \}$, where $n, k > 0$ and $m \geq 0$.

Solution:

(1) Let $\mathtt{l} > +$, $w(\mathtt{l}) = 0$, $w(+) = w(x) > 0$ for each variable x. It holds that w is admissible. For both rules, it holds that the weight does not change after the rewrite step. To prove that $\mathtt{l}(x) + (y + z) >_{\text{kbo}} x + (\mathtt{l}(\mathtt{l}(y)) + z)$ we therefore need to prove the third condition, which holds since $\mathtt{l}(x) >_{\text{kbo}} x$. Similarly, to prove $\mathtt{l}(x) + (y + (z + w)) >_{\text{kbo}} x + (z + (y + w))$, it is sufficient to show that $\mathtt{l}(x) >_{\text{kbo}} x$. Notice that it does not holds $(y + (z + w) >_{\text{kbo}} z + (y + w)$ because of the ordering of the variables.

(2) Let $\mathtt{r} > \mathtt{l}$, $w(\mathtt{r}) = 0$ and $w(\mathtt{l}) = 1$. It holds that w is admissible. Let $s \to t$ with $s = C[\mathtt{r}^n(\mathtt{l}^k(t'))]$ and $t = C[\mathtt{l}^k(\mathtt{r}^m(t'))]$. From the definition of w, it holds that $w(s) = w(t)$ since the number of occurrences of the function symbol $\mathtt{l}$ is the same in s and t. By applying the definition of $>_{\text{kbo}}$, we get that we need to show $\mathtt{r}^n(\mathtt{l}^k(t')) >_{\text{kbo}} \mathtt{l}^k(\mathtt{r}^m(t'))$, which holds since $\mathtt{r} > \mathtt{l}$.

In the following, let $\mathcal{R}$ be a TRS.

A symbol f is **defined** by $\mathcal{R}$ if there is a rule whose left hand-side is headed by f. We will denote with $\mathcal{D}(\mathcal{R})$ the set of all the symbols defined by $\mathcal{R}$.

A **dependency pair** of a rewrite system $\mathcal{R}$ is a pair of terms $(f(\vec{l}), g(\vec{m}))$ where $f(\vec{l}) \to C[g(\vec{m})]$ is a rewrite rule of $\mathcal{R}$ with $g \in \mathcal{D}(\mathcal{R})$.

A **marked dependency pair** of $\mathcal{R}$ is a pair of terms $(f^{\#}(\vec{l}), g^{\#}(\vec{m}))$ such that $(f(\vec{l}), g(\vec{m}))$ is a dependency pair, where $f^{\#}$ is the marked symbol of f.

The **dependency graph** of $\mathcal{R}$ is the directed graph $DG(\mathcal{R})$ defined as follows:

- its nodes are the set of marked dependency pairs;
- there is an edge from (ℓ_1, r_1) to (ℓ_2, r_2) whenever there are substitutions σ_1 and σ_2 such that $r_1\sigma_1 \to^*_{\mathcal{R}} l_2\sigma_2$.

Theorem. A finite term TRS $\mathcal{R}$ is terminating if for each cycle C of $DG(\mathcal{R})$ there is a **weak** reduction order $\geq$ such that

- for every $t \to_R t'$, $t \geq t'$;
- for every $t \to_C t'$, $t > t'$.

Note that we can use a different termination method for each cycle.

The **dependency graph approximation** of $\mathcal{R}$ is the directed graph defined as follows:

- its nodes are the set of marked dependency pairs;
- there is an edge from (ℓ_1, r_1) to (ℓ_2, r_2) if $RC(r_1)$ and ℓ_2 are unifiable.

where $RC(r)$ is the term obtained by replacing with a new variable every strict subterm headed by a defined symbol or a variable, in r.

Interesting read: K. Keiichirou, T. Yoshihito, *On proving AC-termination by AC-dependency pairs.*

WARNING: *this paper is about TRS with Associative and Commutative law. There is however a small introduction on "standard" dependency pairs.*

Exercise 2 :
We consider the following TRS:

$$\begin{aligned} \mathtt{m}(x, \mathtt{0}) &\to 0 & \mathtt{m}(\mathtt{s}(x), \mathtt{s}(y)) &\to \mathtt{m}(x, y) \\ \mathtt{q}(\mathtt{0}, \mathtt{s}(y)) &\to 0 & \mathtt{q}(\mathtt{s}(x), \mathtt{s}(y)) &\to \mathtt{s}(\mathtt{q}(\mathtt{m}(x, y), \mathtt{s}(y))) \\ \mathtt{p}(\mathtt{0}, y) &\to y & \mathtt{p}(x, \mathtt{s}(y)) &\to \mathtt{s}(\mathtt{p}(x, y)) \\ \mathtt{m}(\mathtt{m}(x, y), z) &\to \mathtt{m}(x, \mathtt{p}(y, z)) \end{aligned}$$

1. Which rule makes the termination of this TRS not provable with KBO or RPO?
2. What are the defined symbols?
3. Compute the marked dependency pairs.
4. Draw the dependency graph approximation.
5. What are the inequalities that are enough to consider? What can instead be ignored?
6. Find a weakly monotonic polynomial interpretation on integers satisfying those inequalities.

Solution:

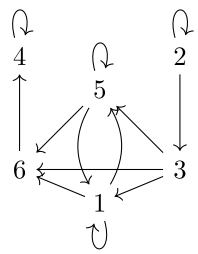
(1) $\mathtt{q}(\mathtt{s}(x), \mathtt{s}(y)) \to \mathtt{s}(\mathtt{q}(\mathtt{m}(x, y), \mathtt{s}(y)))$, since the left-hand side of this rule is embedded in its right-hand side if y is instantiated with $\mathtt{s}(x)$.

(2) The set of defined symbols is $\{\mathtt{m}, \mathtt{q}, \mathtt{p}\}$.

(3) The marked dependency pairs are:

$$
\begin{aligned}
&1 : (\mathtt{m}^{\#}(\mathtt{s}(x), \mathtt{s}(y)),\ \mathtt{m}^{\#}(x, y))\\
&2 : (\mathtt{q}^{\#}(\mathtt{s}(x), \mathtt{s}(y)),\ \mathtt{q}^{\#}(\mathtt{m}(x, y), \mathtt{s}(y)))\\
&3 : (\mathtt{q}^{\#}(\mathtt{s}(x), \mathtt{s}(y)),\ \mathtt{m}^{\#}(x, y))\\
&4 : (\mathtt{p}^{\#}(x, \mathtt{s}(y)),\ \mathtt{p}^{\#}(x, y))\\
&5 : (\mathtt{m}^{\#}(\mathtt{m}(x, y), z),\ \mathtt{m}^{\#}(x, \mathtt{p}(y, z)))\\
&6 : (\mathtt{m}^{\#}(\mathtt{m}(x, y), z),\ \mathtt{p}^{\#}(y, z))
\end{aligned}
$$

(4) Dependency graph approximation:



(5) We need to consider the inequalities:

$$
\begin{aligned}
\mathtt{m}(x, 0) &\geq x\\
\mathtt{m}(\mathtt{s}(x), \mathtt{s}(y)) &\geq \mathtt{m}(x, y)\\
\mathtt{q}(0, \mathtt{s}(y)) &\geq 0\\
\mathtt{q}(\mathtt{s}(x), \mathtt{s}(y)) &\geq \mathtt{s}(\mathtt{q}(\mathtt{m}(x, y), \mathtt{s}(y)))\\
\mathtt{p}(0, y) &\geq y\\
\mathtt{p}(x, \mathtt{s}(y)) &\geq \mathtt{s}(\mathtt{p}(x, y))\\
\mathtt{m}(\mathtt{m}(x, y), z) &\geq \mathtt{m}(x, \mathtt{p}(y, z))
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{m}^{\#}(\mathtt{s}(x), \mathtt{s}(y)) &> \mathtt{m}^{\#}(x, y)\\
\mathtt{q}^{\#}(\mathtt{s}(x), \mathtt{s}(y)) &> \mathtt{q}^{\#}(\mathtt{m}(x, y), \mathtt{s}(y))\\
\mathtt{p}^{\#}(x, \mathtt{s}(y)) &> \mathtt{p}^{\#}(x, y)\\
\mathtt{m}^{\#}(\mathtt{m}(x, y), z) &> \mathtt{m}^{\#}(x, \mathtt{p}(y, z))
\end{aligned}
$$

Whereas the two inequalities that can be ignored are $\mathtt{q}^{\#}(\mathtt{s}(x), \mathtt{s}(y)) > \mathtt{m}^{\#}(x, y)$ and $\mathtt{m}^{\#}(\mathtt{m}(x, y), z) > \mathtt{p}^{\#}(y, z)$, that can be ignored since, looking at the dependency graph approximation, they correspond to nodes that don't belongs to any loops. In fact, we just need to consider, for each strongly connected component, the $\geq$ inequalities plus the $>$ inequalities of that strongly connected component.

(6) In this case, it's easy to find a weakly polynomial interpretation over integers that satisfied all the inequalities, instead of considering separately each strongly connected component: $\mathtt{P}_0 = 0$, $\mathtt{P}_{\mathtt{s}}(X) = X + 2$, $\mathtt{P}_{\mathtt{m}}(X, Y) = X + 1$, $\mathtt{P}_{\mathtt{q}}(X, Y) = 2X$, $\mathtt{P}_{\mathtt{m}^{\#}}(X, Y) = \mathtt{P}_{\mathtt{q}^{\#}}(X, Y) = X$, $\mathtt{P}_{\mathtt{p}}(X, Y) = \mathtt{P}_{\mathtt{p}^{\#}}(X, Y) = X + Y$.

We say that a term is a **minimal non-terminating term** if all its proper subterms are terminating but he is not.

Let C be a cycle in the dependency graph of $\mathcal{R}$ such that every dependency pair symbol in C has positive arity. A **simple projection** for C is a mapping π that assign to every n-ary marked symbol $f^{\#}$ in C an argument position $i \in [1, n]$. We define $\pi(f^{\#}(t_1, \dots, t_n)) = t_{\pi(f^{\#})}$, where $f^{\#}(t_1, \dots, t_n)$ is a term and $f^{\#}$ marked symbol in C.

Theorem. For every non-terminating TRS $\mathcal{R}$ there exists a cycle C in the dependency graph of $\mathcal{R}$ and an infinite rewrite sequence in $\mathcal{R} \cup C$ of the form

$$t_1 \rightarrow^*_{\mathcal{R}} t_2 \rightarrow_C t_3 \rightarrow^*_{\mathcal{R}} t_4 \rightarrow_C t_5 \rightarrow^*_{\mathcal{R}} \dots$$

where $t_1 = f^{\#}(s_1, \dots, s_n)$ is headed by a marked symbol, $f(s_1, \dots, s_n)$ is a minimal non-terminating term and all rules of C are applied infinitely often.

Exercise 3 :

1. Let $\mathcal{R}$ be a rewrite system and such that each defined symbol has positive arity. Prove that if every cycle C of the dependency graph of $\mathcal{R}$ has a simple projection π such that $\pi(C) \subseteq \trianglerighteq$ and $\pi(C) \cap \triangleright \neq \emptyset$, where $\pi(C) = \{(\pi(s), \pi(t)) \mid (s,t) \in C\}$ and $\trianglerighteq$ is the subterm relation, then $\mathcal{R}$ terminates.

Consider the following rewriting system:

$$\begin{array}{rclcrcl}
\mathtt{m}(\mathtt{1}) & \rightarrow & \mathtt{1} & \qquad & \mathtt{m}(\mathtt{a}(x,y)) & \rightarrow & \mathtt{a}(\mathtt{s}(x), \mathtt{m}(y)) \\
\mathtt{q}(\mathtt{0},\mathtt{0}) & \rightarrow & \mathtt{a}(\mathtt{0},\mathtt{1}) & & \mathtt{q}(\mathtt{s}(x),\mathtt{0}) & \rightarrow & \mathtt{1} \\
\mathtt{q}(\mathtt{s}(x),\mathtt{s}(y)) & \rightarrow & \mathtt{m}(\mathtt{q}(x,y)) & & \mathtt{q}(\mathtt{0},\mathtt{s}(y)) & \rightarrow & \mathtt{a}(\mathtt{0},\mathtt{q}(\mathtt{s}(\mathtt{0}),\mathtt{s}(y)))
\end{array}$$

2. Compute the marked dependency pairs and the dependency graph approximation.
3. Prove the termination of the rewrite system by finding a suitable simple projection that satisfied the constraints in question 1.

Solution:

(1) From the Theorem, suppose to the contrary that there exists a rewrite sequence

$$t_1 \rightarrow^*_{\mathcal{R}} u_1 \rightarrow_C t_2 \rightarrow^*_{\mathcal{R}} u_2 \rightarrow_C t_3 \rightarrow^*_{\mathcal{R}} \dots$$

where $t_1 = f^{\#}(s_1, \dots, s_n)$ is headed by a marked symbol, $f(s_1, \dots, s_n)$ is a minimal non-terminating term and all rules of C are applied infinitely often. We apply the simple projection to this rewrite sequence:

- Consider $u_i \rightarrow_C t_{i+1}$. There exists a dependency pair $l \rightarrow r \in C$ and a substitution σ such that $u_i = l\sigma$ and $t_{i+1} = r\sigma$. We have $\pi(u_i) = \pi(l)\sigma$ and $\pi(t_{i+1}) = \pi(r)\sigma$. Since $\pi(l) \rightarrow \pi(r) \in \pi(C)$, by hypothesis it holds $\pi(l) \trianglerighteq \pi(r)$. So $\pi(l) = \pi(r)$ or $\pi(l) \triangleright \pi(r)$. In the former case, trivially $\pi(u_1) = \pi(t_{i+1})$. In the latter case, the closure under substitution of $\triangleright$ yields $\pi(u_i) \triangleright \pi(t_{i+1})$. Because of the assumption $\pi(C) \cap \triangleright \neq \emptyset$, and all rules of C are applied infinitely often, $\pi(u_i) \triangleright \pi(t_{i+1})$ will hold for infinitely many i.
- Consider now $t_i \rightarrow^*_{\mathcal{R}} u_i$. All steps in this sequence take place below the (marked) root symbol, which is therefore the same for t_i and u_i. Therefore $\pi(t_i) \rightarrow^*_{\mathcal{R}} \pi(u_i)$ holds.

By applying our simple projection π to the rewrite sequence, we transform it into a infinite $\rightarrow_{\mathcal{R}} \cup \triangleright$ sequence containing infinitely many $\triangleright$ steps, starting from $\pi(t_1)$. Since $\triangleright$ is well-founded, the sequence must also contain infinitely many $\rightarrow_{\mathcal{R}}$ steps. By making repeated use of the commutation $(\triangleright \rightarrow_{\mathcal{R}}) \subseteq (\rightarrow_{\mathcal{R}} \triangleright)$ we obtain an infinite sequence of $\rightarrow_{\mathcal{R}}$ starting from $\pi(t_1)$. Therefore $\pi(t_1)$ is not terminating w.r.t. $\mathcal{R}$. But $f(s_1, \dots, s_n) \triangleright \pi(t_1)$ and $f(s_1, \dots, s_n)$ is a minimal non-terminating term: contradiction.

(2) The defined symbols are $\{\mathtt{m}, \mathtt{q}\}$. The marked dependency pairs are:

$$\begin{aligned}
1 &: (\mathtt{m}^{\#}(\mathtt{a}(x,y)),\ \mathtt{m}^{\#}(y)) \\
2 &: (\mathtt{q}^{\#}(\mathtt{s}(x),\mathtt{s}(y)),\ \mathtt{m}^{\#}(\mathtt{q}(x,y))) \\
3 &: (\mathtt{q}^{\#}(\mathtt{s}(x),\mathtt{s}(y)),\ \mathtt{q}^{\#}(x,y)) \\
4 &: (\mathtt{q}^{\#}(\mathtt{0},\mathtt{s}(y)),\ \mathtt{q}^{\#}(\mathtt{s}(\mathtt{0}),\mathtt{s}(y)))
\end{aligned}$$

whereas the dependency graph approximation is

$$
\begin{tikzcd}
1 \arrow[loop left] & 2 \arrow[l] \\
3 \arrow[loop left] \arrow[ur] \arrow[r, bend right] & 4 \arrow[u] \arrow[l, bend right]
\end{tikzcd}
$$

There are 3 loops: $\{1\}$, $\{3\}$, $\{3,4\}$. We define $\pi(f^\#) = 1$ and $\pi(g^\#) = 2$. For the first loop it holds $\mathtt{a}(x,y) \rhd y$; for the second loop $\mathtt{s}(y) \rhd y$ and for the last loop $\mathtt{s}(y) \rhd y$ and $\mathtt{s}(y) \unrhd \mathtt{s}(y)$. The conditions to apply the result in question 1 are therefore satisfied and the TRS terminates.

A strict order $>$ on $T(\Sigma, V)$ is called a **rewrite order** if and only iff

1. is *compatible*: for all $s_1, s_2 \in T(\Sigma, V)$, all $f \in \Sigma$, if $s_1 > s_2$ then

$$f(t_1, \ldots, t_{i-1}, s_1, t_{i+1}, \ldots, t_n) > f(t_1, \ldots, t_{i-1}, s_2, t_{i+1}, \ldots, t_n)$$

   where n is the arity of f;

2. is *closed under substitution*: for all $s_1, s_2 \in T(\Sigma, V)$ and all substitutions $\sigma : V \to T(\Sigma, V)$, if $s_1 > s_2$ then $\sigma(s_1) > \sigma(s_2)$.

A strict order $>$ on $T(\Sigma, V)$ satisfies the **subterm property** (and is called *simplification order*) if and only if it is a rewrite order such that for all terms $t \in T(\Sigma, V)$ and all positions $p \in \mathrm{Pos}(t) \setminus \{\epsilon\}$ it holds $t > t|_p$.

Exercise 4 :
In the following, we refer to s, t and w as in the definition of KBO.

1. Assume that f is of arity 1, $w(f) = 0$ and that there is g such that $f \not> g$. Prove that under this conditions $>_{\text{kbo}}$ does not satisfy the subterm property.

Prove that, if w is admissible w.r.t. the strict order $>$ then $>_{\text{kbo}}$ on $T(\Sigma, V)$ induced by $>$ and w has the subterm property. To do so, prove the followings:

2. Assume that $w(s) = w(t)$ and that t is a strict subterm of s. Prove that there exist a unary function f and a positive integer k such that $w(f) = 0$ and $s = f^k(t)$;
3. Prove that $>_{\text{kbo}}$ is a strict order;
4. Prove that $>_{\text{kbo}}$ is a rewrite order;
5. Conclude that $>_{\text{kbo}}$ has the subterm property.

Solution:

(1) Let $t = g(t_1, \ldots, t_n)$ be an arbitrary term with root symbol g, define $s = f(t)$ and let $s \to t$. Since $w(f) = 0$ we have $w(s) = w(t)$. Obviously, the first and third condition of KBO cannot hold, so $>_{\text{kbo}}$ holds if and only if the second condition holds. But this cannot happen since $f \not> g$. As t is a subterm of s, the subterm property is therefore violated.

(2) Proof by induction on the size of s. Since t is a strict subterm of s, there are an $n \geq 1$ and an n-ary function symbol f such that $s = f(s_1, \ldots, s_n)$ and t is a subterm of s_i for some $i \in [1, n]$. First we show that $n = 1$ and $w(f) = 0$:

- Assume $n > 1$. We have $w(s) = w(f) + \sum_{j=1}^{n} w(s_j)$ and (since w admissible) we know that for all j, $w(s_j) \geq w_0 > 0$. Thus, $n > 1$ implies $w(s) > w(s_i)$ and therefore $w(s) > w(t)$. This contradicts the hypothesis $w(s) = w(t)$.
- Assume $w(f) > 0$, then, even for $n = 1$, $w(s) = w(f) + \sum_{j=1}^{n} w(s_j) > w(s_i) \geq w(t)$. This contradicts the hypothesis $w(s) = w(t)$.

This shows that $s = f(s')$ where f is unary, $w(f) = 0$ and s' has t as a subterm. For $s' = t$ we are done. Otherwise we can apply the induction hypothesis since t is a strict subterm of s', $w(s') = w(s) = w(t)$ and $|s'| < |s|$.

(3) **Irreflexivity:** Assume that $>_{\text{kbo}}$ is not *irreflexive*. then s be a term of minimal size such that $s >_{\text{kbo}} s$. Since $w(s) = w(s)$ and the root symbol is the same, then we obtain $s_i >_{\text{kbo}} s_i$ for all $i \in [1, n]$ where n is the arity of the root symbol of s. This contradicts the minimality of s.

To show **transitivity** assume $r >_{\text{kbo}} s$ and $s >_{\text{kbo}} t$, we prove $r >_{\text{kbo}} t$ by induction on the size of r.

- From $r >_{\text{kbo}} s$ and $s >_{\text{kbo}} t$ we deduce that, for all variables x, $|r|_x \geq |s|_x$ and $|s|_x \geq |t|_x$ hold, thus e have $|r|_x \geq |t|_x$. The variables condition is therefore satisfied.
- $r >_{\text{kbo}} s$ and $s >_{\text{kbo}} t$ also yield $w(r) \geq w(s)$ and $w(s) \geq w(t)$, which implies $w(r) \geq w(t)$. Moreover if $w(r) > w(s)$ or $w(s) > w(t)$ then $w(r) > w(t)$ and we are done.

We can assume $w(r) = w(s) = w(t)$. Moreover the second point of the definition cannot hold for $r >_{\text{kbo}} s$, since $s >_{\text{kbo}} t$ implies that s is not a variable. Therefore r and s have a function symbol as root, i.e. $r = f(r_1, \ldots, r_l)$ and $s = g(s_1, \ldots, s_m)$, such that $f \geq g$.

1. If $s >_{\text{kbo}} t$ satisfies the first condition then $t = x$ for a variable x and $|r|_x \geq |t|_x$ implies that x occurs in r. Since the root symbol of r is a function symbol we have $r \neq x$ and from the previous point (2) we have $r >_{\text{kbo}} t$.
2. If instead $s >_{\text{kbo}} t$ satisfies the second or third condition, then we know that there exists a function symbol h such that $g \geq h$ and $t = h(t_1, \ldots, t_n)$. If $f > g$ or $g > h$ then we have $f > h$ and by the second condition $r >_{\text{kbo}} t$. Otherwise, assume $f = g = h$. Then both $r >_{\text{kbo}} s$ and $s >_{\text{kbo}} t$ satisfy the third condition. By induction hypothesis, from the definition of $(>_{\text{kbo}})_{\text{lex}}$ we get $r >_{\text{kbo}} t$.

(4) We first show that $>_{\text{kbo}}$ is **compatible**. Assume $s_1 >_{\text{kbo}} s_2$ and f n-ary function symbol. We must show that the following holds

$$f(t_1, \ldots, t_{i-1}, s_1, t_i, \ldots, t_n) >_{\text{kbo}} f(t_1, \ldots, t_{i-1}, s_2, t_i, \ldots, t_n)$$

From $s_1 >_{\text{kbo}} s_2$ we can deduce that $|s_1|_x \geq |s_2|_x$ for all variables x. This obviously implies

$$|f(t_1, \ldots, t_{i-1}, s_1, t_i, \ldots, t_n)|_x \geq |f(t_1, \ldots, t_{i-1}, s_2, t_i, \ldots, t_n)|_x$$

Moreover, if $w(s_1) > w(s_2)$ then

$$w(f(t_1, \ldots, t_{i-1}, s_1, t_i, \ldots, t_n)) > w(f(t_1, \ldots, t_{i-1}, s_2, t_i, \ldots, t_n))$$

and yields our thesis. Assume instead $w(s_1) = w(s_2)$. This implies that

$$w(f(t_1, \ldots, t_{i-1}, s_1, t_i, \ldots, t_n)) = w(f(t_1, \ldots, t_{i-1}, s_2, t_i, \ldots, t_n))$$

and since the root symbols of the two terms are the same, the thesis holds if and only if the third condition of KBO is satisfied. This is trivial since $t_1 = t_1$,…,$t_{i-1} = t_{i-1}$ and $s_1 >_{\text{kbo}} s_2$.

To show instead that $>_{\text{kbo}}$ is **closed under substitution**, assume $s_1 >_{\text{kbo}} s_2$ and let $\sigma : V \to T(\Sigma, V)$ be a substitution. We show $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$ by induction on the size of s_1. First, consider the variable condition. Let X be the set of variables appearing in s_1. Because of $s_1 >_{\text{kbo}} s_2$ we know that $|s_1|_x \geq |s_2|_x$ for all variables x. For an arbitrary variable x we have

$$|\sigma(s_1)|_x - |\sigma(s_2)|_x = \sum_{y \in X} |\sigma(y)|_x (|s_1|_y - |s_2|_y) \geq 0$$

Thus the variable condition is satisfied. A similar computation can be done for weights:

$$w(\sigma(s_1)) - w(\sigma(s_2)) = w(s_1) - w(s_2) + \sum_{y \in X} (|s_1|_y - |s_2|_y)(w(\sigma(y)) - w_0)$$

For all $y \in X$, it holds $|s_1|_y - |s_2|_y \geq 0$ and $w(\sigma(y)) - w_0 \geq 0$. Consequently $w(s_1) > w(s_2)$ implies $w(\sigma(s_1)) > w(\sigma(s_2))$ which yields $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$. Assume instead that $w(s_1) = w(s_2)$ and hence $w(\sigma(s_1)) \geq w(\sigma(s_2))$.If $w(\sigma(s_1)) > w(\sigma(s_2))$, then $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$. Otherwise we consider one subcase for each condition of KBO:

1. If $s_1 >_{\text{kbo}} s_2$ holds for the first condition, then $s_1 = f^k(x)$ and $s_2 = x$ for a unary symbol f of weight 0 a variable x and a positive integer k. We show $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$ by induction on the size of $\sigma(x)$. If $\sigma(x) = y$ is a variable, then the result trivially holds for the first condition of KBO. Otherwise $\sigma(x) = g(t_1, \dots, t_n)$ for a function symbol g of arity n. If $f \neq g$ then $f > g$ from the admissibility of w and thus $\sigma(s_1) = f^k(g(t_1, \dots, t_n)) >_{\text{kbo}} g(t_1, \dots, t_n) = \sigma(s_2)$ holds from the second condition of KBO. If $f = g$ then the third condition must apply and we need to prove that $f^k(t_1) >_{\text{kbo}} t_1$. By taking a substitution σ' such that $\sigma'(x) = t_1$ then the induction hypothesis (appliable since $\sigma'(x)$ is smaller than $\sigma(x)$) yields $f^k(t_1) = \sigma'(s_1) >_{\text{kbo}} \sigma'(s_2) = t_1$.

2. If $s_1 >_{\text{kbo}} s_2$ holds for the second condition, then the root symbol f of s_1 and the root symbol g of s_2 are such that $f > g$. Obviously $\sigma(s_1)$ has root symbol f whereas $\sigma(s_2)$ as root symbol g, and thus $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$.

3. If $s_2 >_{\text{kbo}} s_2$ holds for the third condition then the root symbols for s_1, s_2, $\sigma(s_1)$ and $\sigma(s_2)$ are the same. Let $s_1 = f(s_1, \dots, s_m)$ and $s_2 = f(t_1, \dots, t_m)$. It holds that there exists $i \in [1, m]$ such that $s_1 = t_1$,…,$s_{i-1} = t_{i-1}$ and $s_i >_{\text{kbo}} t_i$. This implies $\sigma(s_1) = \sigma(t_1)$,…,$\sigma(s_{i-1}) = \sigma(t_{i-1})$ and by induction $\sigma(s_i) >_{\text{kbo}} \sigma(t_i)$ (since s_i is smaller than s_1). Thus $\sigma(s_1) > \sigma(s_2)$ holds.

(5) To show the subterm property, recall that $s >_{\text{kbo}} x$ for all variables x and terms $s \neq x$ that contain x. This, together with the fact that $>_{\text{kbo}}$ is closed under substitutions, obviously implies the subterm property.