Rewriting Techniques: TD 2

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Exercise 1 :
Let A and B be two terminating relations on the same set of terms. Show that if $AB \subseteq BA$ then $A \cup B$ is a terminating relation. Use well-founded induction.

Solution:

We prove that $(A \cup B)^* \subseteq B^*A^*$ by well-founded induction on steps of $(A \cup B)^*$. The induction hypothesis is: $t(\to_{A\cup B})^n t' \implies t \to_B^* \to_A^* t'$. W.l.o.g. suppose $t \to_{A\cup B}^{n+1} t'$ such that $t \to_A t'' \to_B (\to_{A\cup B})^{n-1} t'$ for some $n \in \mathbb{N}$. Then by hypothesis $AB \subseteq BA$ it holds that there exists t''' such that $t \to_B t''' \to_A (\to_{A\cup B})^{n-1} t'$. By inductive hypothesis, since $t'''(\to_{A\cup B})^n t'$, $t''' \to_B^* \to_A^* t'$ and therefore $t \to_B^* \to_A^* t'$. Since A and B are terminating relations, B^*A^* is a terminating relation. It follows that $A \cup B$ is terminating.

Given a strict order $>$ on a set A, we define the corresponding **multiset order** $>_{\mathrm{mul}}$ on $\mathtt{Mult}(A)$ as follows: $M >_{\mathrm{mul}} N$ if and only if there exist $X, Y \in \mathtt{Mult}(A)$ such that

1. $\emptyset \neq X \subseteq M$;
2. $N = (M \setminus X) \cup Y$;
3. $\forall y \in Y \ \exists x \in X \ x > y$.

Exercise 2 :
Prove that $M >_{\mathrm{mul}} N$ if and only if $M \neq N$ and for all $n \in N \setminus M$ there exists $m \in M \setminus N$ such that $m > n$.

Solution:

Assume $M >_{\mathrm{mul}} N$, X and Y as in the definition. We first show that $N \setminus M = Y \setminus X$ and $M \setminus N = X \setminus Y$.

- $N \setminus M = ((M \setminus X) \cup Y) \setminus M = ((M \cup Y) \setminus X) \setminus M = ((M \cup Y) \setminus M) \setminus X = Y \setminus X$.
  Here, the second equality holds because $X \subseteq M$ *(moreover, recall that we are working with multisets and not sets! Otherwise,* $(M \setminus X) \cup Y \neq (M \cup Y) \setminus X$*).*
- $M \setminus N = M \setminus ((M \setminus X) \cup Y) = (M \setminus (M \setminus X)) \setminus Y = X \setminus Y$.
  Here, the last equality holds as $X \subseteq M$. Indeed, $M \setminus (M \setminus X)$ is then

$$\lambda x.(M(x) - (M(x) - X(x))).$$

($\Rightarrow$) $M \neq N$ follows from irreflexivity of $>_{\mathrm{mul}}$. For the 2nd condition, suppose $y_1 \in N \setminus M = Y \setminus X$. By definition, there is a $y_2 \in X$ such that $y_2 > y_1$. Either $y_2 \in X \setminus Y = M \setminus N$, in which case we are done, or $y_2 \in X \cap Y$ (where $(X \cap Y)(x) = \min(X(x), Y(x))$), in which case there is $y_3 \in X$ such that $y_3 > y_2$. Because our multisets are finite and $>$ is a strict order, there is no infinite ascending chain $y_1 < y_2 < \dots$ in $X \cap Y$. This process therefore terminate with some $y_n \in X \setminus Y = M \setminus N$. Transitivity yields $y_n > y_1$.

($\Leftarrow$) Let $N \setminus M = Y$ and $M \setminus N = X$. Since $M \neq N$, it cannot be that M and N are both empty. Moreover it holds that $X \neq \emptyset$. Indeed, suppose that $X = M \setminus N$ is empty, then for all e $M(e) \leq N(e)$ and since $M \neq N$ it will hold that $N \neq \emptyset$, which will lead $N \setminus M$ not empty. This is contradictory, as we assumed $M \setminus N = \emptyset$ but, from the second condition we have that $N \setminus M \neq \emptyset$ implies $M \setminus N \neq \emptyset$. Moreover from it's definition $X \subseteq M$. By definition it also holds that $N = (M \setminus X) \cup Y$. Indeed, $N = (N \cap M) \cup (N \setminus M)$. We can easily show that $N \cap M = (M \setminus (M \setminus N))$. Then by definition of X and Y we obtain $N = (M \setminus X) \cup Y$. From the last hypothesis it holds that for all $n \in Y$ there exists $m \in X$ such that $m > n$. We conclude $M >_{\text{mul}} N$.

The **lexicographic order** $>_{\text{lex}}$ for the Cartesian product $\times$ of two domains $(A, >_A)$ and $(B, >_B)$ is defined as follows: $(a_1, b_1) >_{\text{lex}} (a_2, b_2)$ if and only if $a_1 > a_2$ or $a_1 = a_2$ and $b_1 > b_2$. This order can be readily extended con Cartesian products of arbitrary length by recursively applying this definition, i.e by observing that $A \times B \times C = A \times (B \times C)$.

In the following, let Σ be a finite signature, V a set of variables and $T(\Sigma, V)$ the terms built from those sets.

For every $f \in \Sigma$ let $\texttt{status}(f) \in \{\text{mul}, \text{lex}\}$ be its *status* ($\texttt{status}$ is then called status function) and let $>$ be a strict order on Σ. The **recursive path order** $>_{\text{rpo}}$ on $T(\Sigma, V)$ induced by $>$ is defined as follows. $s >_{\text{rpo}} t$ if and only if one of the following holds:

1. t is a variable appearing in s and $s \neq t$, or

let $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$,

2. there exists $i \in [1, m]$ such that $s_i \geq_{\text{rpo}} t$, or
3. $f > g$ and $s >_{\text{rpo}} t_j$ for all $j \in [1, n]$, or
4. $f = g$, for all $j \in [1, m]$ it holds $s >_{\text{rpo}} t_j$ and $(s_1, \ldots, s_m)(>_{\text{rpo}})_{\texttt{status}(f)}(t_1, \ldots, t_m)$.

The **lexicographic path order** is a recursive path order s.t. for all $f \in \Sigma$, $\texttt{status}(f) = \text{lex}$, whereas the **multiset path order** is a recursive path order s.t. for all $f \in \Sigma$, $\texttt{status}(f) = \text{mul}$, where we define $(s_1, \ldots, s_m)(>_{\text{rpo}})_{\text{mul}}(t_1, \ldots, t_m)$ as $\{\!\!\{ s_1, \ldots, s_m \}\!\!\}(>_{\text{rpo}})_{\text{mul}}\{\!\!\{ t_1, \ldots, t_m \}\!\!\}$.

Exercise 3 :
We consider the Ackermann's function

$$\begin{aligned}
\texttt{Ack}\ 0\ y &= y + 1 \\
\texttt{Ack}\ x\ 0 &= \texttt{Ack}\ (x - 1)\ 1 \\
\texttt{Ack}\ x\ y &= \texttt{Ack}\ (x - 1)\ (\texttt{Ack}\ x\ (y - 1))
\end{aligned}$$

1. Prove its termination via well-founded induction.
2. The following rewrite system simulates $\texttt{Ack}$

$$\texttt{a}(\texttt{0}, y) \to \texttt{s}(y)$$
$$\texttt{a}(\texttt{s}(x), \texttt{0}) \to \texttt{a}(x, \texttt{s}(\texttt{0}))$$
$$\texttt{a}(\texttt{s}(x), \texttt{s}(y)) \to \texttt{a}(x, \texttt{a}(\texttt{s}(x), y))$$

   Prove its termination using a RPO.
3. Consider the well-founded domain $(\texttt{Mult}(\mathbb{N} \times \mathbb{N}), (>_{\text{lex}})_{\text{mul}})$. Prove the termination of $\texttt{Ack}$ using the following abstraction:

$$\begin{aligned}
\phi :\ T(\{\texttt{a}, \texttt{s}\}, X) &\to \texttt{Mult}(\mathbb{N} \times \mathbb{N}) \\
t &\to \{\!\!\{ (|u|, |v|) \mid t|_{p \in \text{Pos}(t)} = \texttt{a}(u, v) \}\!\!\}
\end{aligned}$$

   where $|\texttt{0}| = 1$, $|\texttt{a}(x, y)| = |x| + |y| + 1$ and $|\texttt{s}(x)| = |x| + 1$.

Solution:

(1) Induction on $(\mathbb{N} \times \mathbb{N}, >_{\text{lex}})$. We prove that the calculus of $\texttt{Ack}\ u\ v$ terminates by induction (u, v) ordered lexicographically on integers.

- Base cases: Ack terminates for $(0, n)$, $n \in \mathbb{N}$;
- We need to show that Ack terminates for (n, m), $n > 0$. Induction hypothesis: Ack terminates for all (j, k) such that $j < n$ or ($j = n$ and $k < m$). If $m = 0$ then by induction hypothesis the function terminates since $(n-1, 1) < (n, m)$. Instead, if $m > 0$, by induction hypothesis the function terminates on input $(n, m-1)$ with output r and terminates on input $(n-1, r)$.

(2) Let $>$ be such that $\mathtt{a} > \mathtt{s}$ and let $\mathtt{status}(\mathtt{a}) = \mathtt{status}(\mathtt{s}) = \text{lex}$. It holds:

- $\mathtt{a}(0, t) >_{\text{rpo}} \mathtt{s}(t)$, since $\mathtt{a} > \mathtt{s}$ and $\mathtt{a}(0, t) >_{\text{rpo}} t$;
- $\mathtt{a}(\mathtt{s}(t), 0) >_{\text{rpo}} \mathtt{a}(t, \mathtt{s}(0))$ since $\mathtt{a}(\mathtt{s}(t), 0) > t$, $\mathtt{a}(\mathtt{s}(t), 0) > \mathtt{s}(0)$ and $(\mathtt{s}(t), 0)(>_{\text{rpo}})_{\text{lex}}(t, \mathtt{s}(0))$;
- $\mathtt{a}(\mathtt{s}(t), \mathtt{s}(t')) >_{\text{rpo}} \mathtt{a}(t, \mathtt{a}(\mathtt{s}(t), t'))$ since

$$\mathtt{a}(\mathtt{s}(t), \mathtt{s}(t')) >_{\text{rpo}} t,\ \ \mathtt{a}(\mathtt{s}(t), \mathtt{s}(t')) >_{\text{rpo}} \mathtt{a}(\mathtt{s}(t), t'),\ \ (\mathtt{s}(t), \mathtt{s}(t'))(>_{\text{rpo}})_{\text{lex}}(t, \mathtt{a}(\mathtt{s}(t), t'))$$

(3) Let $s \to t$,

- if $s = C[\mathtt{a}(0, t')]$ and $t = C[\mathtt{s}(t')]$, then $\phi(t) = \phi(s) \setminus \{\!\!|\ (1, |t'|)\ |\!\!\}$. Therefore it holds

$$\phi(s)(>_{\text{lex}})_{\text{mul}}\phi(t).$$

- if $s = C[\mathtt{a}(\mathtt{s}(t'), 0)]$ and $t = C[\mathtt{a}(\mathtt{s}(t'), \mathtt{s}(0))]$, then

$$\phi(t) = \phi(s) \cup \{\!\!|\ (|t'|, 2)\ |\!\!\} \setminus \{\!\!|\ (|t'| + 1, 1)\ |\!\!\}.$$

- if $s = C[\mathtt{a}(\mathtt{s}(t'), \mathtt{s}(t''))]$ and $t = C[\mathtt{a}(t', \mathtt{a}(\mathtt{s}(t'), t''))]$, then

$$\phi(t) = \phi(s) \cup \{\!\!|\ (|t'|, |t'| + |t''| + 2), (|t'| + 1, |t''|)\ |\!\!\} \setminus \{\!\!|\ (|t'| + 1, |t''| + 1)\ |\!\!\}.$$

Exercise 4 :
Show that is not possible to prove termination using lexicographic path ordering of the following term rewrite system:

$$\{\ \mathtt{a}(\mathtt{a}(x)) \to \mathtt{s}(x),\ \ \mathtt{s}(\mathtt{s}(x)) \to \mathtt{a}(x)\ \}$$

Solution:

Suppose $\mathtt{a} > \mathtt{s}$. Let $s \to t$ with $s = C[\mathtt{s}(\mathtt{s}(t')]$ and $t = C[\mathtt{a}(t')]$. Then we need to prove that $\mathtt{s}(\mathtt{s}(t')) >_{\text{rpo}} \mathtt{a}(t')$, which does not hold. If instead $\mathtt{s} > \mathtt{a}$, then from the first rule we get $\mathtt{a}(\mathtt{a}(t')) >_{\text{rpo}} \mathtt{s}(t')$ which, again, does not hold.

Exercise 5 :
Show that the termination of the following rewriting system cannot be proven with lexicographic path order but can be proven with multiset path order.

$$\mathtt{0} + x \to \mathtt{0} \qquad\qquad \mathtt{0} \times x \to x$$
$$\mathtt{s}(x) + y \to \mathtt{s}(x + y) \qquad\qquad \mathtt{s}(x) \times y \to (y \times x) + y$$

Solution:

We need to impose the precedence $\times > + > \mathtt{s}$. With this, the lexicographic path order cannot orient the last rule as it does not hold that $(\mathtt{s}(x), y) >_{\text{lex}} (y, x)$. Instead, it holds $\{\!\!|\ \mathtt{s}(x), y\ |\!\!\} >_{\text{mul}} \{\!\!|\ y, x\ |\!\!\}$ since $\mathtt{s}(x)$ dominates x.