Exercise 1:
Given the following term rewriting system (TRS):

\[
\begin{align*}
x \times 0 & \rightarrow 0 \\
0 \times x & \rightarrow 0 \\
s(x) \times y & \rightarrow (x \times y) + y \\
x \times s(y) & \rightarrow (x \times y) + x \\
0 + 0 & \rightarrow 0 \\
x + 0 & \rightarrow x \\
0 + x & \rightarrow x \\
x + s(y) & \rightarrow s(x + y) \\
s(x) + y & \rightarrow s(x + y)
\end{align*}
\]

Show the reduction graph of \(((0 \times 0) + 0) + s(0)\).

Solution:

Exercise 2:
Given the signature \(\{\mathbb{N}, \text{List}\}, \{0, s, \epsilon, :, \text{merge}, \text{sort}\}\) where the set of functions is typed as follows:

\[
0 : \mathbb{N}, \quad s : \mathbb{N} \rightarrow \mathbb{N}, \quad \epsilon : \text{List}, \quad (:) : \mathbb{N} \times \text{List} \rightarrow \text{List}, \\
\text{merge} : \text{List} \times \text{List} \rightarrow \text{List}, \quad \text{sort} : \text{List} \rightarrow \text{List}
\]

Define a finite TRS that simulates the \textit{mergesort algorithm}. If needed, you can define auxiliary sorts and function symbols.

Solution:

We will use the additional sort \(\mathbb{B} = \{\top, \bot\}\) and the following function symbols:

\[
\begin{align*}
\text{even} : \text{List} & \rightarrow \text{List}, \quad \text{odd} : \text{List} \rightarrow \text{List}, \\
\geq : \mathbb{N} \times \mathbb{N} & \rightarrow \mathbb{B}, \quad \text{aux} : \mathbb{N} \times \text{List} \times \text{List} \rightarrow \text{List}
\end{align*}
\]

We define the following TRS:

\[
\begin{align*}
\text{even}(\epsilon) & \rightarrow \epsilon \\
\text{odd}(\epsilon) & \rightarrow \epsilon \\
\text{even}(x:\epsilon) & \rightarrow \epsilon \\
\text{odd}(x:\epsilon) & \rightarrow x:\epsilon \\
\text{even}(x:y:z) & \rightarrow y: \text{even}(z) \\
\text{odd}(x:y:z) & \rightarrow x: \text{odd}(z) \\
0 \geq 0 & \rightarrow \top \\
\text{aux}(\top, x:y:z:w) & \rightarrow z: \text{merge}(x:y, w) \\
\text{aux}(\bot, x:y:z:w) & \rightarrow x: \text{merge}(y, z:w) \\
0 \geq s(x) & \rightarrow \bot \\
s(x) \geq 0 & \rightarrow \top \\
s(x) \geq s(y) & \rightarrow x \geq y
\end{align*}
\]
merge(x, ε) → x
merge(ε, x) → x
merge(x, y, z, w) → aux(x ≥ z, x, y, z, w)

\( \text{sort}(\epsilon) \rightarrow \epsilon \)
\( \text{sort}(x: \epsilon) \rightarrow x: \epsilon \)
\( \text{sort}(x, y, z) \rightarrow \text{merge} (\text{sort} (\text{even}(x, y, z)), \text{sort} (\text{odd}(x, y, z))) \)

Exercise 3:
Let \( M = (\Sigma, Q, \Delta) \) be a non-deterministic Turing machine where

- \( \Sigma = \{s_0, \ldots, s_n\} \) is a finite alphabet and \( s_0 \) is considered the blank symbol;
- \( Q = \{q_0, \ldots, q_p\} \) is a finite set of states;
- \( \Delta \subseteq Q \times \Sigma \times \Sigma \times \{l, r\} \) transition relation.

A configuration is an ordered triple \( (x, q, k) \in \Sigma^{*} \times Q \times \mathbb{N} \) where \( x \) denotes the string on the tape, \( q \) denotes the machine’s current state, and \( k \) denotes the position of the machine on the tape.

Translate \( M \) into a finite TRS such that there exists an injection \( f \) from configurations of \( M \) to terms satisfying for each configuration \( \gamma, \gamma' \):

\[
\begin{align*}
\gamma & \Delta \gamma' \\
\gamma & \rightarrow \gamma'
\end{align*}
\]

\[
\begin{align*}
f(\gamma) & \rightarrow f(\gamma) \\
f(\gamma') & \rightarrow f(\gamma')
\end{align*}
\]

\[
\begin{align*}
f^{-1}(\gamma) & \rightarrow f^{-1}(\gamma) \\
f^{-1}(\gamma') & \rightarrow f^{-1}(\gamma')
\end{align*}
\]

Solution:
For each \( s \in \Sigma \) we introduce the unary function symbol \( s \). For each \( q \in Q \) we introduce the unary function symbol \( q \). Lastly, we introduce the constant \( r \) and the unary function symbol \( 1 \).

We now define the TRS:

- For each transition \( (q, s_i, q', s_j, r) \in \Delta \) we add the rewriting rule \( q(s_i(x)) \rightarrow s_j(q'(x)) \) and if \( i = 0 \) we also add the rule \( q(x) \rightarrow s_j(q'(r)) \).
- For each transition \( (q, s_i, q', s_j, l) \in \Delta \) we add the rule \( l(q(s_i(x))) \rightarrow l(q'(s_0(s_j(x)))) \) and for every \( k \in [1, n] \) we add the rewriting rule \( s_k(q(s_i(x))) \rightarrow q'(s_k(s_j(x))) \). Moreover, if \( i = 0 \), we also add \( l(q(r)) \rightarrow l(q'(s_0(s_j(x)))) \) and the rule \( s_k(q(r)) \rightarrow q'(s_k(s_j(x))) \), where \( k \in [1, n] \).

Finally, for a configuration \( (x, q, k) \) where \( x = s_{i_0}s_{i_1}\ldots s_{i_k-1}s_{i_k}s_{i_k+1}\ldots s_{i_t} \), the injection \( f \) is defined as \( l(s_{i_0}(s_{i_1}(\ldots s_{i_k-1}(q(s_{i_k}(s_{i_k+1}(\ldots s_{i_t}(r)))))))) \).

Exercise 4:
Are the following TRS terminating?

1. \{ \( s(p(x)) \rightarrow x, p(s(x)) \rightarrow x \) \};
2. \{ \( s(p(x)) \rightarrow x, p(s(x)) \rightarrow s(p(x)) \) \};
3. \{ \( s(p(x)) \rightarrow x, p(s(x)) \rightarrow s(s(p(x))) \) \};

For each transition system, let \( t \) and \( t' \) be two terms with the same normal form. What is the relationship between \( t \) and \( t' \)?
A polynomial interpretation on integers is the following:

- a subset $A$ of $\mathbb{N}$;
- for every symbol $f$ of arity $n$, a polynomial $p_f \in \mathbb{N}[X_1, \ldots, X_n]$;
- for every $a_1, \ldots, a_n \in A$, $p_f(a_1, \ldots, a_n) \in A$;
- for every $a_1, \ldots, a_i > a'_i, \ldots, a_n \in A$, $p_f(a_1, \ldots, a_i, \ldots, a_n) > p_f(a_1, \ldots, a'_i, \ldots, a_n)$;

Then $(A, (p_f)_f, >)$ is a well-founded monotone algebra.

**Exercise 5:**
Prove the termination of the following TRS

$$
\begin{align*}
0 \times x & \rightarrow 0 \\
\text{s}(x) \times y & \rightarrow (x \times y) + y \\
x + 0 & \rightarrow x \\
x + \text{s}(y) & \rightarrow \text{s}(x + y)
\end{align*}
$$

using the polynomial interpretation on integers:

$$p_0 = 2 \quad p_s(X) = X + 1 \quad p_p(X, Y) = X + 2Y \quad p_x(X, Y) = (X + Y)^2$$

Is this polynomial interpretation suitable to prove termination of the TRS of Exercise 1?

**Solution:**

(1) From the polynomial interpretation we get the following polynomial for the various rules of the TRS: $p_{0 \times x}(X) = (X + 2)^2$, $p_{\text{s}(x) \times y}(X, Y) = (X + Y + 1)^2$, $p_{(x \times y) + y}(X, Y) = (X + Y)^2 + 2Y$, $p_x(X) = X + 4$, $p_{x + s(y)}(X, Y) = X + 2(Y + 1)$ and $p_{s(x + y)} = X + 2Y + 1$.

- $p_{0 \times x}(X) > p_0$ true since $(X + 2)^2 = X^2 + 4X + 4 > 2$;
- $p_{\text{s}(x) \times y}(X, Y) > p_{(x \times y) + y}(X, Y)$ true since $(X + Y + 1)^2 = X^2 + 2XY + Y^2 + 2X + 2Y + 1$ is greater than $(X + Y)^2 + 2Y = X^2 + 2XY + Y^2 + 2Y$;
- $p_x(X) > X$ true since $X + 4 > X$;
- $p_{x + s(y)}(X, Y) > p_{s(x + y)}$ since $X + 2(Y + 1) > X + 2Y + 1$.

(2) No. For the rule $\text{s}(x) + y \rightarrow \text{s}(x + y)$. Indeed, $p_{\text{s}(x) + y}(X, Y) = p_{\text{s}(x + y)}(X, Y) = X + 2Y + 1$.

**Exercise 6:**
Prove the termination of the following TRS by finding a polynomial interpretation on integers:

$$
\begin{align*}
x \times (y + z) & \rightarrow (x \times y) + (x \times z) \\
(x + y) + z & \rightarrow x + (y + z)
\end{align*}
$$
Solution:
Let $P_\times(X, Y)$ and $P_+(X, Y)$ be the two polynomial interpretation that we want to find. We can start by showing that the polynomial interpretation for the second rule must have degree 1. Let $\deg_{\times}(X)$ be the degree of the polynomial $P_\times(X, Y)$ w.r.t. the variable $X$. Similarly we denote with $\deg_{\times}(Y)$ the degree of the polynomial $P_\times(X, Y)$ w.r.t. $Y$, whereas $\deg_{+}(X)$ and $\deg_{+}(Y)$ are the degrees of the polynomial $P_+(X, Y)$ w.r.t. $X$ and $Y$ respectively. From the first rule it must hold that $\deg_{\times}(X) \geq \deg_{\times}(X) \times \deg_{+}(X)$, which implies $\deg_{+}(X) = 1$. Moreover it holds $\deg_{\times}(X) \geq \deg_{\times}(X) \times \deg_{+}(Y)$, which implies $\deg_{+}(Y) = 1$. Therefore, $P_+(X, Y)$ must be of the form $s_2X + s_1Y + s_0$. From the second rule we obtain

$$s_2(s_2X + s_1Y + s_0) + s_1Z + s_0 > s_2X + s_1(s_2Y + s_1Z + s_0) + s_0$$

Which can be rewritten as $s_2^2X + s_1Z + s_0s_2 > s_2X + s_1^2Z + s_0s_1$. It follows that $s_2$ must be greater than $s_1$. With a similar reasoning it follows that $P_\times(X, Y)$ must have degree 2.

Let’s define the polynomial interpretation on $\mathbb{N} \setminus \{0, 1\}$, $P_\times(X, Y) = XY$ and $P_+(X, Y) = 2X + Y + 1$. For the first rule, the left side of the rule is interpreted with $X(2Y + Z + 1)$ whereas the right side is $2XY + XZ + 1$. It holds $2XY + XZ + 1 > 2XY + XZ + 1$ whenever $X > 1$ (and for this reason we use an interpretation on $\mathbb{N} \setminus \{0, 1\}$). Similarly, for the second one it holds that $4X + 2Y + Z + 3 > 2X + 2Y + Z + 2$.

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**A polynomial interpretation on real numbers** is the following:

- a subset $A$ of $\mathbb{R}^+$;
- a positive real number $\delta$;
- for every symbol $f$ of arity $n$, a polynomial $P_f \in \mathbb{R}[X_1, \ldots, X_n]$;
- for every $a_1, \ldots, a_n \in A$, $P_f(a_1, \ldots, a_n) \in A$;
- for every $a_1, \ldots, a_i > \delta a'_i, \ldots, a_n \in A$, $P_f(a_1, \ldots, a_i, \ldots, a_n) > \delta P_f(a_1, \ldots, a'_i, \ldots, a_n)$ where $x > \delta y$ iff $x > y + \delta$.

Then $(A, (P_f), > \delta)$ is a well-founded monotone algebra.

**Exercise 7:**
Consider the following two TRS:

$$R_1 = \{ p(1(x)) \rightarrow p(p(1(x))), \ s(a(x)) \rightarrow s(s(a(x))), \ p(x) \rightarrow a(x, x), \ s(x) \rightarrow a(x, 0), \ s(x) \rightarrow a(0, x) \}$$

$$R_2 = \{ r(r(r(x))) \rightarrow a(r(x), r(x)), \ s(a(r(x), r(x))) \rightarrow r(r(r(x))) \}$$

1. Prove that $R_1 \cup R_2$ terminates using the following polynomial interpretation on real numbers: $\delta = 1$, $P_0(X) = 0$, $P_1(X) = X^2$, $P_2(X) = X + 4$, $P_3(X) = 3X + 5$, $P_4(X, Y) = X + Y$ and $P_5(X) = \sqrt{2}X + 1$.
2. Prove that in any polynomial interpretation on integers proving the termination of $R_1$ it must hold that $P_2(X)$ is of the form $X + s_0$ and $P_4(X, Y)$ is of the form $X + Y + a_0$, with $s_0 > a_0$.

**hint:** look at the dominant terms of the polynomials computed from the rewrite rules.
3. Deduce that the termination of $R_1 \cup R_2$ cannot be proved using a polynomial interpretation of integers.
Solution:

(1) \[
P_{\ell(p(z)))}(X) = 9X^2 + 30X + 25 >_1 P_{\ell(p(1z)))}(X) = 9X^2 + 20
\]
\[
P_{\ell(p(z)))}(X) = 3X + 17 >_1 P_{\ell(s(p(z)))}(X) = 3X + 13
\]
\[
P_{\ell(p(z)))}(X) = 3X + 5 >_1 P_{\ell(s,z,x))(X) = 2X
\]
\[
P_{\ell(s,z))(X) = X + 4 >_1 P_{\ell(s,0,z))(X) = X
\]
\[
P_{\ell(s,z,x))(X) = 2\sqrt{2}X + 3 + \sqrt{2} >_1 P_{\ell(v,(z)))}(X) = 2\sqrt{2}X + 2
\]
\[
P_{\ell(s,v,(z)))}(X) = 2\sqrt{2}X + 6 >_1 P_{\ell(v,(z)))}(X) = 2\sqrt{2}X + 3 + \sqrt{2}
\]

(2) Let \( P_0 = z \geq 0 \). From the second rule of \( R_1 \), let \( \alpha \) be the degree of \( P_a(X) \) and let \( \beta \) be the degree of \( P_b(X) \). From \( P_{a(x)}(X) > P_{a(p(x)))}(X) \) it must hold that \( \beta \alpha \geq \alpha \alpha \beta \). Therefore \( \alpha = 1 \). Similarly, from the first rule, also \( P_b(X) \) is of degree one. From the third rule it must hold that \( P_a(X, Y) \) is also of degree one. So \( P_b(X) \) is of the form \( p_1X + p_0 \), \( P_b(X) \) is of the form \( s_1X + s_0 \) whereas \( P_a(X, Y) \) is of the form \( a_2X + a_1Y + a_0 \). From the fourth rule it must hold \( s_1 + s_0 > a_2X + a_0 + a_1z \), which implies \( s_1 \geq a_2 \geq 1 \). Similarly, from the fifth rule, \( s_1 \geq a_1 \geq 1 \). From the second rule \( s_1p_1 + s_0p_1 + p_0 > s_1^2p_1 + s_1^2p_0 + s_1s_0 + s_0 \) and therefore it must hold that \( s_1p_1 \geq s_1^2p_1 \). Therefore \( s_1 = 1 \), which also implies \( a_2 = a_1 = 1 \). Moreover from \( s_1 + s_0 > a_2X + a_0 + a_1z \), it must hold \( s_0 > a_0 \).

(3) Let \( \alpha \) be the degree of the polynomial \( P(X) \). From the second rule of \( R_2 \) it must hold that \( \alpha^3 \leq \alpha \) and therefore \( \alpha = 1 \) and \( P(X) \) is of the form \( r_1X + r_0 \). Looking now at the first rule, it must hold that \( r_1(r_1X + r_0) + r_0 > 2r_1X + 2r_0 + a_0 \) which implies \( r_1^2 \geq 2r_1 \) and therefore \( r_1^2 \geq 2 \). Similarly, from the second rule of \( R_2 \) it must hold that \( 2r_1 \geq r_1^3 \) or alternatively \( r_1^2 \leq 2 \). Therefore \( r_1^2 \) must be equal to \( 2 \), which requires \( r_1 = \sqrt{2} \) not to be a natural number.

A matrix interpretation on integers is the following:

- a positive integer \( d \);
- for every symbol \( f \) of arity \( n \), \( n \) matrices \( M_{f,1}, \ldots, M_{f,n} \in \mathbb{N}^{d \times d} \);
- for every symbol of arity \( n \), a vector \( V_f \in \mathbb{N}^d \);
- a non-empty set \( I \subseteq \{1, \ldots, d\} \) satisfying that for every symbol \( f \) of arity \( n \) the map

\[
L_f : (\mathbb{N}^d)^n \to \mathbb{N}^d \text{ defined as } L_f(X_1, \ldots, X_n) = V_f + \sum_{i=1}^n M_{f,i}X_i
\]

is monotonic with respect to \( >_I \) were \( X >_I Y \) holds if and only if for every \( i \in \{1, \ldots, d\} \), \( X[i] > Y[i] \) and there is \( j \in I \) such that \( X[j] > Y[j] \).

Then \( (\mathbb{N}^d, (L_f)_f, >_I) \) is a well-founded monotone algebra.

**Exercise 8:**

Consider the TRS \{ \( s(a) \to s(p(a)), p(b) \to p(s(b)) \) \}.

1. Prove that its termination cannot be proved by a polynomial interpretation on integers;

2. Use the following matrix interpretation to prove termination w.r.t. \( >_{\{1,2\}} \):

   \[
   L_a(X) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} X \quad L_p(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X \quad L_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad L_b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
   \]

3. Why does it fail if we take \( >_{\{1\}} \) instead? Is there another matrix interpretation that works with this ordering?
Deduce that the TRS simulating the Ackermann’s function, cannot be proved terminating using a polynomial interpretation.

Consider now any finite TRS and a function symbol of a polynomial interpretation of integers for a TRS.

Let \( A \) be respectively the domain and the interpretation, for each function symbol \( f \), of a polynomial interpretation of integers for a TRS (note: the TRS is therefore terminating). Take \( a \in A \setminus \{0\} \).

1. Define \( \pi_a : T(F,X) \rightarrow A \) as the function which maps every variable \( x \) to \( a \) and every term of the form \( f(t_1, \ldots, t_n) \) to \( P_f(\pi_a(t_1), \ldots, \pi_a(t_n)) \). Prove that \( \pi_a(t) \) is greater or equal to the length of every reduction starting from \( t \).

2. Show that there exists \( d \) and \( k \) positive integers such that for every \( f \in F \) of arity \( n \) and every \( a_1, \ldots, a_n \in A \setminus \{0\} \) it holds \( P_f(a_1, \ldots, a_n) \leq d \prod_{i=1}^n a_i^k \).

3. From the previous point, pick \( d \) to be also greater or equal than \( a \) and fix \( c \geq k + \log_2(d) \). Prove that \( \pi_a(t) \leq 2^{2^{|t|}} \).

Consider now any finite TRS and a function symbol \( f \). Prove that there exists an integer \( k \) such that if \( s \rightarrow t \) then \(|t|_f \leq k(|s|_f + 1) \), where \(|.|_f \) is the number of \( f \).

Deduce that the TRS
\[
\{ a(0, y) \rightarrow s(y), \ a(s(x), 0) \rightarrow a(x, s(0)), \ a(s(x), s(y)) \rightarrow a(x, a(s(x), y)) \}
\]
simulating the Ackermann’s function, cannot be proved terminating using a polynomial interpretation over integers.

Solution:

(1) The proof is by induction on the \( \rightarrow \) relation. Let \( t \) be irreducible. Then the length of all its reductions is 0 and \( \pi_a(t) \geq 0 \) by definition. For the inductive step, suppose \( t \rightarrow t’ \) s.t. \( t \rightarrow t’ \ldots \) is the maximal reduction from \( t \). There exists a context \( C \), a valuation \( \sigma \) and a rewriting rule \( l \rightarrow r \) such that \( t = C[l\sigma] \rightarrow C[r\sigma] = t’ \). Without loss of generality, we can consider just terms of the form \( l\sigma \rightarrow r\sigma \). Let \( P_f \) and \( P_r \) be the polynomials resulting from the polynomial interpretation, for \( l \) and \( r \) respectively. We have that, for all \( X_1, \ldots, X_n, P_f(X_1, \ldots, X_n) > P_r(X_1, \ldots, X_n) \).
inductive hypothesis, \( \pi_a(r\sigma) = P_f(\pi_a(\sigma(X_1)), \ldots, \pi_a(\sigma(X_n))) \) is greater or equal to the length of every reduction starting from \( r\sigma \). It follow that \( \pi_a(l\sigma) = P_f(\pi_a(\sigma(X_1)), \ldots, \pi_a(\sigma(X_n))) \geq \pi_a(r\sigma) + 1 \) and therefore \( \pi_a(l\sigma) \) is greater or equal to the length of every reduction starting from \( l\sigma \).

(2) Let \( \{s_0, \ldots, s_m\} \) be the coefficient of the polynomial \( P_f \), let \( d \geq \sum_{i=0}^{n} s_i \) (so \( d \geq 1 \)) and let \( k \geq 1 \) be also greater or equal to the degree of \( P_f \). The thesis can be rewritten as \( P_f(a_1, \ldots, a_n) \leq (\sum_{i=1}^{m} s_i) \prod_{j=1}^{n} a_j^k = \sum_{i=1}^{m} (s_i \prod_{j=1}^{n} a_j^k) \). Moreover there exists

\[
k_{1,1}, \ldots, k_{1,n}, k_{2,1}, \ldots, k_{2,n}, \ldots, k_{m,1}, \ldots, k_{m,n}
\]

such that \( P_f(a_1, \ldots, a_n) = \sum_{i=1}^{m} (s_i \prod_{j=1}^{n} a_j^{k_{i,j}}) \) and for all \( i \in [1, m] \) \( k_{1,1} + \cdots + k_{i,n} \leq k \). Moreover \( a_1, \ldots, a_n \in A \setminus \{0\} \), and therefore the thesis trivially holds since for all \( i \in [1, m] \) \( s_i \prod_{j=1}^{n} a_j^{k_{i,j}} \leq s_i \prod_{j=1}^{n} a_j^k \).

(3) By induction of \( t \). If \( t \) is a variable, then \( |t| = 1 \) and \( \pi_a(t) = a \leq 2^k \leq 2^{2\log_2(d)} \leq 2^{2^c} \). If \( t \) is of the form \( f(t_1, \ldots, t_n) \) then \( \pi_a(t) = P_f(\pi_a(t_1), \ldots, \pi_a(t_n)) \). By inductive hypothesis, since \( P_f \) is monotone, \( \pi_a(t) \leq P_f(2^{2^c}, \ldots, 2^{2^c}) \). From (2) it follows that \( P_f(2^{2^c}, \ldots, 2^{2^c}) \leq d \prod_{i=1}^{n} (2^{2^c})^{k} = d 2^{\sum_{i=1}^{n} (2^{2^c})^{k}} = 2^{d \log_2(d) + k(2^{2^c})} \leq 2^{(d \log_2(d) + k)(2^{2^c})} \). Since \( d \geq a \geq 1 \) and \( k \geq 1 \) it holds that \( c \geq 1 \) and therefore \( 2^{d \log_2(d) + k)(2^{2^c})} \leq 2^{2^{(d + k)(2^{2^c})}} \leq 2^{2^c} \).

(4) W.l.o.g. consider \( s = l\sigma \) and \( t = r\sigma \) for a rewriting rule \( l \rightarrow r \) and a valuation \( \sigma \). The number of occurrences of \( f \) in \( l\sigma \) is \( |l|_f + \sum_{p \in \{p|\sigma(p) \in X\}} |\sigma(p)|_f \) where \( |l|_f \) only depends on the left side of the rewriting rule. Similarly, \( |r\sigma|_f = |r|_f + \sum_{p \in \{p|\sigma(p) \in X\}} |\sigma(p)|_f \) where \( |r|_f \) depends only on the right side of the rewriting rule. Let \( V \) the number of variables in \( r \) (i.e. \( \{p|\sigma(p) \in X\} \)). It holds that \( |r|_f \leq |r|_f + V \max_{p \in \{p|\sigma(p) \in X\}} |\sigma(p)|_f \). Since every variable of \( r \) also occurs in \( l \) it must hold that \( |r|_f \leq |r|_f + V \max_{p \in \{p|\sigma(p) \in X\}} |\sigma(p)|_f \). Moreover \( \max_{p \in \{p|\sigma(p) \in X\}} |\sigma(p)|_f \) is trivially less or equal that all the occurrences of \( f \) in \( l\sigma \), therefore \( |r|_f \leq |r|_f + V |\sigma|_f \leq (|r|_f + V)(|\sigma|_f + 1) \). |\sigma|_f and \( V \) only depends on the rule itself. Let \( k \) be greater or equal than the maximum number of occurrences of \( f \) in the right side of each rule of the TRS plus the number of variables in the right side of each rule of the TRS. It holds that \( |r|_f \leq k(|\sigma|_f + 1) \).

(5) From the above point, it holds that for all terms \( s \) and \( t \) such that \( s \rightarrow t \), \( |t|_a \leq k(|s|_a + 1) \). So at each step of the rewriting system, the number of \( a \) can at most increase \( k \) times (from the previous proof, for Ackermann this should hold for \( k \geq 5 \)). If Ackermann could be proved terminating using a polynomial interpretation over integers then given any term \( t \), the maximum number of steps will be \( \pi_a(t) \leq 2^{2^c} \), where \( c \) is fixed (and depends on the polynomial interpretation, see proof (2)). The size of a term of the form \( a(m, n) \) is \( m + n + 3 \). We conclude that there must exists \( k \) and \( c \) such that for any \( X, Y \) it should hold \( Ack(X, Y) \leq k \cdot 2^{2^c} \). This cannot hold since \( Ack(X, Y) \) is not primitive recursive whereas \( k \cdot 2^{2^c} \) is, and therefore there exists \( X, Y \) such that \( Ack(X, Y) > k \cdot 2^{2^c} \).