Rewriting Techniques: TD 1

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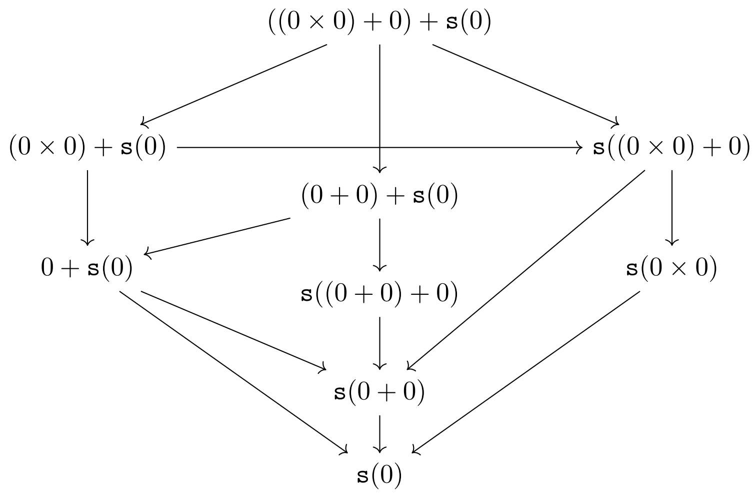
Exercise 1 :
Given the following term rewriting system (TRS):

$$x \times \mathtt{0} \to \mathtt{0} \qquad x + \mathtt{0} \to x$$
$$\mathtt{0} \times x \to \mathtt{0} \qquad \mathtt{0} + x \to x$$
$$\mathtt{s}(x) \times y \to (x \times y) + y \qquad x + \mathtt{s}(y) \to \mathtt{s}(x + y)$$
$$x \times \mathtt{s}(y) \to (x \times y) + x \qquad \mathtt{s}(x) + y \to \mathtt{s}(x + y)$$

Show the *reduction graph* of $((0 \times 0) + 0) + \mathtt{s}(0)$.

Solution:



Exercise 2 :
Given the signature $(\{\mathbb{N}, \mathtt{List}\}, \{\mathtt{0}, \mathtt{s}, \epsilon, :, \mathtt{merge}, \mathtt{sort}\})$ where the set of functions is typed as follows:

$$\mathtt{0} : \mathbb{N}, \qquad \mathtt{s} : \mathbb{N} \to \mathbb{N}, \qquad \epsilon : \mathtt{List}, \qquad (:) : \mathbb{N} \times \mathtt{List} \to \mathtt{List},$$
$$\mathtt{merge} : \mathtt{List} \times \mathtt{List} \to \mathtt{List}, \qquad \mathtt{sort} : \mathtt{List} \to \mathtt{List}$$

Define a finite TRS that simulates the *mergesort algorithm*. If needed, you can define auxiliary sorts and function symbols.

Solution:

We will use the additional sort $\mathbb{B} = \{\top, \bot\}$ and the following function symbols:

$\mathtt{even} : \mathtt{List} \to \mathtt{List}$, $\mathtt{odd} : \mathtt{List} \to \mathtt{List}$, $\geq : \mathbb{N} \times \mathbb{N} \to \mathbb{B}$, $\mathtt{aux} : \mathbb{N} \times \mathtt{List} \times \mathtt{List} \to \mathtt{List}$

We define the following TRS:

$$\mathtt{even}(\epsilon) \to \epsilon \qquad \mathtt{odd}(\epsilon) \to \epsilon$$
$$\mathtt{even}(x{:}\epsilon) \to \epsilon \qquad \mathtt{odd}(x{:}\epsilon) \to x{:}\epsilon$$
$$\mathtt{even}(x{:}y{:}z) \to y{:}\mathtt{even}(z) \qquad \mathtt{odd}(x{:}y{:}z) \to x{:}\mathtt{odd}(z)$$

$$\mathtt{0} \geq \mathtt{0} \to \top \qquad \mathtt{aux}(\top, x{:}y, z{:}w) \to z{:}\mathtt{merge}(x{:}y, w)$$
$$\mathtt{s}(x) \geq \mathtt{0} \to \top \qquad \mathtt{aux}(\bot, x{:}y, z{:}w) \to x{:}\mathtt{merge}(y, z{:}w)$$
$$\mathtt{0} \geq \mathtt{s}(x) \to \bot$$
$$\mathtt{s}(x) \geq \mathtt{s}(y) \to x \geq y$$

$\texttt{merge}(x, \epsilon) \to x$
$\texttt{merge}(\epsilon, x) \to x$
$\texttt{merge}(x{:}y, z{:}w) \to \texttt{aux}(x \geq z, x{:}y, z{:}w)$

$\texttt{sort}(\epsilon) \to \epsilon$
$\texttt{sort}(x{:}\epsilon) \to x{:}\epsilon$
$\texttt{sort}(x{:}y{:}z) \to \texttt{merge}(\texttt{sort}(\texttt{even}(x{:}y{:}z)), \texttt{sort}(\texttt{odd}(x{:}y{:}z)))$

Exercise 3 :
Let $\mathcal{M} = (\Sigma, Q, \Delta)$ be a non-deterministic Turing machine where

- $\Sigma = \{s_0, \ldots, s_n\}$ is a finite alphabet and s_0 is considered the blank symbol;
- $Q = \{q_0, \ldots, q_p\}$ is a finite set of states;
- $\Delta \subseteq Q \times \Sigma \times Q \times \Sigma \times \{l, r\}$ transition relation.

A configuration is an ordered triple $(x, q, k) \in \Sigma^* \times Q \times \mathbb{N}$ where x denotes the string on the tape, q denotes the machine's current state, and k denotes the position of the machine on the tape. Translate $\mathcal{M}$ into a finite TRS such that there exists an injection $\mathfrak{f}$ from configurations of $\mathcal{M}$ to terms satisfying for each configuration γ, γ'.

$$\begin{tikzcd} \gamma \arrow[r, "\Delta"] \arrow[d, "\mathfrak{f}"'] & \gamma' \arrow[d, "\mathfrak{f}"] \\ \mathfrak{f}(\gamma) \arrow[r] & \mathfrak{f}(\gamma') \end{tikzcd} \qquad \begin{tikzcd} \gamma \arrow[r, "\Delta"] & \gamma' \\ \mathfrak{f}(\gamma) \arrow[r] \arrow[u, "\mathfrak{f}^{-1}"] & \mathfrak{f}(\gamma') \arrow[u, "\mathfrak{f}^{-1}"'] \end{tikzcd}$$

Solution:

For each $s \in \Sigma$ we introduce the unary function symbol $\texttt{s}$. For each $q \in Q$ we introduce the unary function symbol $\texttt{q}$. Lastly, we introduce the constant $\texttt{r}$ and the unary function symbol $\texttt{l}$.

We now define the TRS:

- For each transition $(q, s_i, q', s_j, r) \in \Delta$ we add the rewriting rule $\texttt{q}(\texttt{s}_i(x)) \to \texttt{s}_j(\texttt{q}'(x))$ and if $i = 0$ we also add the rule $\texttt{q}(\texttt{r}) \to \texttt{s}_j(\texttt{q}'(\texttt{r}))$.
- For each transition $(q, s_i, q', s_j, l) \in \Delta$ we add the rule $\texttt{l}(\texttt{q}(\texttt{s}_i(x))) \to \texttt{l}(\texttt{q}'(\texttt{s}_0(\texttt{s}_j(x))))$ and for every $k \in [1, n]$ we add the rewriting rule $\texttt{s}_k(\texttt{q}(\texttt{s}_i(x))) \to \texttt{q}'(\texttt{s}_k(\texttt{s}_j(x)))$. Moreover, if $i = 0$, we also add $\texttt{l}(\texttt{q}(\texttt{r})) \to \texttt{l}(\texttt{q}'(\texttt{s}_0(\texttt{s}_j(\texttt{r}))))$ and the rule $\texttt{s}_k(\texttt{q}(\texttt{r})) \to \texttt{q}'(\texttt{s}_k(\texttt{s}_j(x)))$, where $k \in [1, n]$.

Finally, for a configuration (x, q, k) where $x = s_{i_0} s_{i_1} \ldots s_{i_{k-1}} s_{i_k} s_{i_{k+1}} \ldots s_{i_\ell}$, the injection $\mathfrak{f}$ is defined as $\texttt{l}(\texttt{s}_{i_0}(\texttt{s}_{i_1}(\ldots \texttt{s}_{i_{k-1}}(\texttt{q}(\texttt{s}_k(\texttt{s}_{i_{k+1}}(\ldots \texttt{s}_{i_\ell}(\texttt{r}))))))))$.

Exercise 4 :
Are the following TRS terminating?

1. $\{\ \texttt{s}(\texttt{p}(x)) \to x,\ \texttt{p}(\texttt{s}(x)) \to x\ \}$;
2. $\{\ \texttt{s}(\texttt{p}(x)) \to x,\ \texttt{p}(\texttt{s}(x)) \to \texttt{s}(\texttt{p}(x))\ \}$;
3. $\{\ \texttt{s}(\texttt{p}(x)) \to x,\ \texttt{p}(\texttt{s}(x)) \to \texttt{s}(\texttt{s}(\texttt{p}(\texttt{p}(x))))\ \}$;

For each transition system, let t and t' be two terms with the same normal form. What is the relationship between t and t'?

Solution:

(1) Terminates since the number of symbols is always decreasing. Use the polynomial interpretation on natural numbers $\mathtt{P}_{\mathtt{s}}(X) = \mathtt{P}_{\mathtt{p}}(X) = X + 1$.

(2) Also terminates. Use the polynomial interpretation on natural numbers $\mathtt{P}_{\mathtt{s}}(X) = X + 2$ and $\mathtt{P}_{\mathtt{p}}(X) = X^2$.

(3) Does not terminate. Show the reduction graph of $\mathtt{p}(\mathtt{s}(\mathtt{s}(\mathtt{0})))$.

Let $t|_{\mathtt{s}}$ and $t|_{\mathtt{p}}$ be respectively the number of occurrences of the $\mathtt{s}$ and $\mathtt{p}$ in the term t. Two terms t and t' have the same normal form if and only if $t|_{\mathtt{s}} - t|_{\mathtt{p}} = t'|_{\mathtt{s}} - t'|_{\mathtt{p}}$. Moreover, if $t|_{\mathtt{s}} \geq t|_{\mathtt{p}}$, their normal form is $\mathtt{s}^{t|_{\mathtt{s}} - t|_{\mathtt{p}}}(\mathtt{0})$, otherwise it's $\mathtt{p}^{t|_{\mathtt{p}} - t|_{\mathtt{s}}}(\mathtt{0})$.

A **polynomial interpretation on integers** is the following:

- a subset A of $\mathbb{N}$;
- for every symbol f of arity n, a polynomial $\mathtt{P}_f \in \mathbb{N}[X_1, \dots, X_n]$;
- for every $a_1, \dots, a_n \in A$, $\mathtt{P}_f(a_1, \dots, a_n) \in A$;
- for every $a_1, \dots, a_i > a_i', \dots, a_n \in A$, $\mathtt{P}_f(a_1, \dots, a_i, \dots, a_n) > \mathtt{P}_f(a_1, \dots, a_i', \dots, a_n)$;

Then $(A, (\mathtt{P}_f)_f, >)$ is a well-founded monotone algebra.

Exercise 5 :
Prove the termination of the following TRS

$$\mathtt{0} \times x \to \mathtt{0} \qquad\qquad x + \mathtt{0} \to x$$
$$\mathtt{s}(x) \times y \to (x \times y) + y \qquad\qquad x + \mathtt{s}(y) \to \mathtt{s}(x + y)$$

using the polynomial interpretation on integers:

$$\mathtt{P}_0 = 2 \qquad \mathtt{P}_{\mathtt{s}}(X) = X + 1 \qquad \mathtt{P}_+(X, Y) = X + 2Y \qquad \mathtt{P}_\times(X, Y) = (X + Y)^2$$

Is this polynomial interpretation suitable to prove termination of the TRS of Exercise 1?

Solution:

(1) From the polynomial interpretation we get the following polynomial for the various rules of the TRS: $\mathtt{P}_{0\times x}(X) = (X+2)^2$, $\mathtt{P}_{\mathtt{s}(x)\times y}(X, Y) = (X+Y+1)^2$, $\mathtt{P}_{(x\times y)+y}(X, Y) = (X+Y)^2+2Y$, $\mathtt{P}_{x+0}(X) = X + 4$, $\mathtt{P}_{x+\mathtt{s}(y)}(X, Y) = X + 2(Y + 1)$ and $\mathtt{P}_{\mathtt{s}(x+y)} = X + 2Y + 1$.

- $\mathtt{P}_{0\times x}(X) > \mathtt{P}_0$ true since $(X + 2)^2 = X^2 + 4X + 4 > 2$;
- $\mathtt{P}_{\mathtt{s}(x)\times y}(X, Y) > \mathtt{P}_{(x\times y)+y}(X, Y)$ true since $(X+Y+1)^2 = X^2+2XY+Y^2+2X+2Y+1$ is greater than $(X + Y)^2 + 2Y = X^2 + 2XY + Y^2 + 2Y$;
- $\mathtt{P}_{x+0}(X) > X$ true since $X + 4 > X$;
- $\mathtt{P}_{x+\mathtt{s}(y)}(X, Y) > \mathtt{P}_{\mathtt{s}(x+y)}$ since $X + 2(Y + 1) > X + 2Y + 1$.

(2) No. For the rule $\mathtt{s}(x)+y \to \mathtt{s}(x+y)$. Indeed, $\mathtt{P}_{\mathtt{s}(x)+y}(X, Y) = \mathtt{P}_{\mathtt{s}(x+y)}(X, Y) = X+2Y+1$.

Exercise 6 :
Prove the termination of the following TRS by finding a polynomial interpretation on integers:

$$x \times (y + z) \to (x \times y) + (x \times z)$$
$$(x + y) + z \to x + (y + z)$$

Solution:

Let $\mathtt{P}_\times(X,Y)$ and $\mathtt{P}_+(X,Y)$ be the two polynomial interpretation that we want to find. We can start by showing that the polynomial interpretation for the second rule must have degree 1. Let $\deg_{\times(X)}$ be the degree of the polynomial $\mathtt{P}_\times(X,Y)$ w.r.t. the variable X. Similarly we denote with $\deg_{\times(Y)}$ the degree of the polynomial $\mathtt{P}_\times(X,Y)$ w.r.t. Y, whereas $\deg_{+(X)}$ and $\deg_{+(Y)}$ are the degrees of the polynomial $\mathtt{P}_+(X,Y)$ w.r.t. X and Y respectively. From the first rule it must hold that $\deg_{\times(X)} \geq \deg_{\times(X)} \times \deg_{+(X)}$, which implies $\deg_{+(X)} = 1$. Moreover it holds $\deg_{\times(X)} \geq \deg_{\times(X)} \times \deg_{+(Y)}$, which implies $\deg_{+(Y)} = 1$. Therefore, $\mathtt{P}_+(X,Y)$ must be of the form $s_2X + s_1Y + s_0$. From the second rule we obtain

$$s_2(s_2X + s_1Y + s_0) + s_1Z + s_0 > s_2X + s_1(s_2Y + s_1Z + s_0) + s_0$$

Which can be rewritten as $s_2^2X + s_1Z + s_0s_2 > s_2X + s_1^2Z + s_0s_1$. It follows that s_2 must be greater than s_1. With a similar reasoning it follows that $\mathtt{P}_\times(X,Y)$ must have degree 2.

Lets define the polynomial interpretation on $\mathbb{N} \setminus \{0,1\}$, $\mathtt{P}_\times(X,Y) = XY$ and $\mathtt{P}_+(X,Y) = 2X + Y + 1$. For the first rule, the left side of the rule is interpreted with $X(2Y + Z + 1)$ whereas the right side is $2XY + XZ + 1$. It holds $2XY + XZ + X > 2XY + XZ + 1$ whenever $X > 1$ (and for this reason we use an interpretation on $\mathbb{N} \setminus \{0,1\}$). Similarly, for the second one it holds that $4X + 2Y + Z + 3 > 2X + 2Y + Z + 2$.

A **polynomial interpretation on real numbers** is the following:

- a subset A of $\mathbb{R}^+$;
- a positive real number δ;
- for every symbol f of arity n, a polynomial $\mathtt{P}_f \in \mathbb{R}[X_1, \ldots, X_n]$;
- for every $a_1, \ldots, a_n \in A$, $\mathtt{P}_f(a_1, \ldots, a_n) \in A$;
- for every $a_1, \ldots, a_i >_\delta a_i', \ldots, a_n \in A$, $\mathtt{P}_f(a_1, \ldots, a_i, \ldots, a_n) >_\delta \mathtt{P}_f(a_1, \ldots, a_i', \ldots, a_n)$ where $x >_\delta y$ iff $x > y + \delta$.

Then $(A, (\mathtt{P}_f)_f, >_\delta)$ is a well-founded monotone algebra.

Exercise 7 :
Consider the following two TRS:

$$\begin{aligned} R_1 =\{\ & \mathtt{l}(\mathtt{p}(x)) \to \mathtt{p}(\mathtt{p}(\mathtt{l}(x))),\ \mathtt{p}(\mathtt{s}(x)) \to \mathtt{s}(\mathtt{s}(\mathtt{p}(x))),\ \mathtt{p}(x) \to \mathtt{a}(x,x), \\ & \mathtt{s}(x) \to \mathtt{a}(x,\mathtt{0}),\ \mathtt{s}(x) \to \mathtt{a}(\mathtt{0},x)\ \} \\ R_2 =\{\ & \mathtt{r}(\mathtt{r}(\mathtt{r}(x))) \to \mathtt{a}(\mathtt{r}(x),\mathtt{r}(x)),\ \mathtt{s}(\mathtt{a}(\mathtt{r}(x),\mathtt{r}(x))) \to \mathtt{r}(\mathtt{r}(\mathtt{r}(x)))\ \} \end{aligned}$$

1. Prove that $R_1 \cup R_2$ terminates using the following polynomial interpretation on real numbers: $\delta = 1$, $\mathtt{P}_\mathtt{0}(X) = 0$, $\mathtt{P}_\mathtt{l}(X) = X^2$, $\mathtt{P}_\mathtt{s}(X) = X + 4$, $\mathtt{P}_\mathtt{p}(X) = 3X + 5$, $\mathtt{P}_\mathtt{a}(X,Y) = X + Y$ and $\mathtt{P}_\mathtt{r}(X) = \sqrt{2}X + 1$.
2. Prove that in any polynomial interpretation on integers proving the termination of R_1 it must hold that $\mathtt{P}_\mathtt{s}(X)$ is of the form $X + s_0$ and $\mathtt{P}_\mathtt{a}(X,Y)$ is of the form $X + Y + a_0$, with $s_0 > a_0$.
   hint: look at the dominant terms of the polynomials computed from the rewrite rules.
3. Deduce that the termination of $R_1 \cup R_2$ cannot be proved using a polynomial interpretation of integers.

Solution:
(1)

$$\begin{aligned}
\mathtt{P}_{\mathtt{1}(\mathtt{p}(x))}(X) = 9X^2 + 30X + 25 >_1 \mathtt{P}_{\mathtt{p}(\mathtt{p}(\mathtt{1}(x)))}(X) = 9X^2 + 20 \\
\mathtt{P}_{\mathtt{p}(\mathtt{s}(x))}(X) = 3X + 17 >_1 \mathtt{P}_{\mathtt{s}(\mathtt{s}(\mathtt{p}(x)))}(X) = 3X + 13 \\
\mathtt{P}_{\mathtt{p}(x)}(X) = 3X + 5 >_1 \mathtt{P}_{\mathtt{a}(x,x)}(X) = 2X \\
\mathtt{P}_{\mathtt{s}(x)}(X) = X + 4 >_1 \mathtt{P}_{\mathtt{a}(x,0)}(X) = X \\
\mathtt{P}_{\mathtt{s}(x)}(X) = X + 4 >_1 \mathtt{P}_{\mathtt{a}(0,x)} = X \\
\mathtt{P}_{\mathtt{r}(\mathtt{r}(\mathtt{r}(x)))}(X) = 2\sqrt{2}X + 3 + \sqrt{2} >_1 \mathtt{P}_{\mathtt{a}(\mathtt{r}(x),\mathtt{r}(x))} = 2\sqrt{2}X + 2 \\
\mathtt{P}_{\mathtt{s}(\mathtt{a}(\mathtt{r}(x),\mathtt{r}(x)))}(X) = 2\sqrt{2}X + 6 >_1 \mathtt{P}_{\mathtt{r}(\mathtt{r}(\mathtt{r}(x)))}(X) = 2\sqrt{2}X + 3 + \sqrt{2}
\end{aligned}$$

(2) Let $P_0 = z \geq 0$. From the second rule of R_1, let α be the degree of $\mathtt{P}_\mathtt{s}(X)$ and let β be the degree of $\mathtt{P}_\mathtt{p}(X)$. From $\mathtt{P}_{\mathtt{ps}(x)}(X) > \mathtt{P}_{\mathtt{s}(\mathtt{s}(\mathtt{p}(x)))}(X)$ it must hold that $\beta\alpha \geq \alpha\alpha\beta$. Therefore $\alpha = 1$. Similarly, from the first rule, also $\mathtt{P}_\mathtt{p}(X)$ is of degree one. From the third rule it must hold that $\mathtt{P}_\mathtt{a}(X, Y)$ is also of degree one. So $\mathtt{P}_\mathtt{p}(X)$ is of the form $p_1X + p_0$, $\mathtt{P}_\mathtt{s}(X)$ is of the form $s_1X + s_0$ whereas $\mathtt{P}_\mathtt{a}(X, Y)$ is of the form $a_2X + a_1Y + a_0$. From the fourth rule it must hold $s_1X + s_0 > a_2X + a_0 + a_1z$, which implies $s_1 \geq a_2 \geq 1$. Similarly, from the fifth rule, $s_1 \geq a_1 \geq 1$. From the second rule $s_1p_1X + s_0p_1 + p_0 > s_1^2p_1X + s_1^2p_0 + s_1s_0 + s_0$ and therefore it must hold that $s_1p_1 \geq s_1^2p_1$. Therefore $s_1 = 1$, which also implies $a_2 = a_1 = 1$. Moreover from $s_1X + s_0 > a_2X + a_0 + a_1z$, it must hold $s_0 > a_0$.
(3) Let α be the degree of the polynomial $\mathtt{P}_\mathtt{r}(X)$. From the second rule of R_2 it must hold that $\alpha^3 \leq \alpha$ and therefore $\alpha = 1$ and $\mathtt{P}_\mathtt{r}(X)$ is of the form $r_1X + r_0$. Looking now at the first rule, it must hold that $r_1(r_1(r_1X + r_0) + r_0) + r_0 > 2r_1X + 2r_0 + a_0$ which implies $r_1^3 \geq 2r_1$ and therefore $r_1^2 \geq 2$. Similarly, from the second rule of R_2 it must hold that $2r_1 \geq r_1^3$ or alternatively $r_1^2 \leq 2$. Therefore r_1^2 must be equal to 2, which requires r_1 $(= \sqrt{2})$ not to be a natural number.

A **matrix interpretation on integers** is the following:

- a positive integer d;
- for every symbol f of arity n, n matrices $M_{f,1} \ldots, M_{f,n} \in \mathbb{N}^{d\times d}$;
- for every symbol of arity n, a vector $V_f \in \mathbb{N}^d$;
- a non-empty set $I \subseteq \{1, \ldots, d\}$ satisfying that for every symbol f of arity n the map

  $$L_f : (\mathbb{N}^d)^n \to \mathbb{N}^d \text{ defined as } L_f(X_1, \ldots, X_n) = V_f + \sum_{i=1}^{n} M_{f,i}X_i$$

  is monotonic with respect to $>_I$ were $X >_I Y$ holds if and only if for every $i \in \{1, \ldots, d\}$, $X[i] \geq Y[i]$ and there is $j \in I$ such that $X[j] > Y[j]$.

Then $(\mathbb{N}^d, (L_f)_f, >_I)$ is a well-founded monotone algebra.

Exercise 8 :
Consider the TRS $\{$ $\mathtt{s(a)} \to \mathtt{s(p(a))}$, $\mathtt{p(b)} \to \mathtt{p(s(b))}$ $\}$.

1. Prove that its termination cannot be proved by a polynomial interpretation on integers;
2. Use the following matrix interpretation to prove termination w.r.t. $>_{\{1,2\}}$.

$$L_\mathtt{s}(X) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} X \qquad L_\mathtt{p}(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X \qquad L_\mathtt{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad L_\mathtt{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3. Why does it fail if we take $>_{\{1\}}$ instead? Is there another matrix interpretation that works with this ordering?

Solution:

(1) From the first rule, the degree of $\mathtt{P_p}$ must be one. The same holds for $\mathtt{P_s}$, thanks to the second rule. This implies that $\mathtt{P_{s(a)}}$ of the form $s_1 a + s_0$ whereas $\mathtt{P_{s(p(a))}}$ of the form $s_1 p_1 a + s_1 p_0 + s_0$ which, since $p_1 \geq 1$, is sufficient to conclude that the termination of this TRS cannot be proved by a polynomial interpretation on integers.

(2) It holds that:

$$L_{\mathtt{s}}(L_{\mathtt{a}}) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} >_{\{1,2\}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = L_{\mathtt{s}}(L_{\mathtt{p}}(L_{\mathtt{a}}))$$

$$L_{\mathtt{p}}(L_{\mathtt{b}}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} >_{\{1,2\}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = L_{\mathtt{p}}(L_{\mathtt{s}}(L_{\mathtt{b}}))$$

(3) From the second rule, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} >_{\{1\}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not hold. No, let $L_{\mathtt{s}}(X) = M_{\mathtt{s}}X + V_{\mathtt{s}}$ and $L_{\mathtt{p}}(X) = M_{\mathtt{p}}X + V_{\mathtt{p}}$. For the first rule it will hold

$$L_{\mathtt{s}}(L_{\mathtt{a}}) = M_{\mathtt{s}}L_{\mathtt{a}} + V_{\mathtt{s}}$$
$$L_{\mathtt{s}}(L_{\mathtt{p}}(L_{\mathtt{a}})) = M_{\mathtt{s}}M_{\mathtt{p}}L_{\mathtt{a}} + M_{\mathtt{s}}V_{\mathtt{p}} + V_{\mathtt{s}}$$

To make $>_{\{1\}}$ it must therefore hold

$$(M_{\mathtt{s}})_{1,1}(L_{\mathtt{a}})_{1,1} + \dots + (M_{\mathtt{s}})_{1,d}(L_{\mathtt{a}})_{1,d} > (M_{\mathtt{p}})_{1,1}((M_{\mathtt{s}})_{1,1}(L_{\mathtt{a}})_{1,1} + \dots + (M_{\mathtt{s}})_{1,d}(L_{\mathtt{a}})_{1,d}) + \dots$$

which implies $(M_{\mathtt{p}})_{1,1} = 0$. Similarly, from the second rule, $(M_{\mathtt{s}})_{1,1} = 0$. This implies that no polynomial interpretation with the ordering $>_{\{1\}}$ can be defined for this TRS, since for any $m > n$, $(m, 0, \dots, 0) >_{\{1\}} (n, 0, \dots, 0)$ but $L_{\mathtt{s}}(m, 0, \dots, 0) =_{\{1\}} L_{\mathtt{s}}(n, 0, \dots, 0)$.

Exercise 9 :

Let $A \subseteq \mathbb{N}$ and $\mathtt{P}_f$ be respectively the domain and the interpretation, for each function symbol f, of a polynomial interpretation of integers for a TRS *(note: the TRS is therefore terminating)*. Take $a \in A \setminus \{0\}$.

1. Define $\pi_a : T(F, X) \to A$ as the function which maps every variable x to a and every term of the form $f(t_1, \dots, t_n)$ to $\mathtt{P}_f(\pi_a(t_1), \dots, \pi_a(t_n))$. Prove that $\pi_a(t)$ is greater or equal to the length of every reduction starting from t.
2. Show that there exists d and k positive integers such that for every $f \in F$ of arity n and every $a_1, \dots, a_n \in A \setminus \{0\}$ it holds $\mathtt{P}_f(a_1, \dots, a_n) \leq d \prod_{i=1}^{n} a_i^k$.
3. From the previous point, pick d to be also greater or equal than a and fix $c \geq k + \log_2(d)$. Prove that $\pi_a(t) \leq 2^{2^{c|t|}}$.

Consider now any finite TRS and a function symbol f. Prove that there exists an integer k such that if $s \to t$ then $|t|_f \leq k(|s|_f + 1)$, where $|.|_f$ is the number of f.

Deduce that the TRS

$$\{\ \mathtt{a}(\mathtt{0}, y) \to \mathtt{s}(y),\ \mathtt{a}(\mathtt{s}(x), \mathtt{0}) \to \mathtt{a}(x, \mathtt{s}(\mathtt{0})),\ \mathtt{a}(\mathtt{s}(x), \mathtt{s}(y)) \to \mathtt{a}(x, \mathtt{a}(\mathtt{s}(x), y))\ \},$$

simulating the Ackermann's function, cannot be proved terminating using a polynomial interpretation over integers.

Solution:

(1) The proof is by induction on the $\to$ relation. Let t be irreducible. Then the length of all its reductions is 0 and $\pi_a(t) \geq 0$ by definition. For the inductive step, suppose $t \to t'$ s.t. $t \to t' \to \dots$ is the maximal reduction from t. There exists a context C, a valuation σ and a rewriting rule $l \to r$ such that $t = C[l\sigma] \to C[r\sigma] = t'$. W.l.o.g. we can consider just terms of the form $l\sigma \to r\sigma$. Let $\mathtt{P}_l$ and $\mathtt{P}_r$ be the polynomials resulting from the polynomial interpretation, for l and r respectively. We have that, for all $X_1, \dots, X_n$, $\mathtt{P}_l(X_1, \dots, X_n) > \mathtt{P}_r(X_1, \dots, X_n)$. By

inductive hypothesis, $\pi_a(r\sigma) = \mathtt{P}_r(\pi_a(\sigma(X_1)), \ldots, \pi_a(\sigma(X_n)))$ is greater or equal to the length of every reduction starting from $r\sigma$. It follow that $\pi_a(l\sigma) = \mathtt{P}_l(\pi_a(\sigma(X_1)), \ldots, \pi_a(\sigma(X_n))) \geq \pi_a(r\sigma) + 1$ and therefore $\pi_a(l\sigma)$ is greater or equal to the length of every reduction starting from $l\sigma$.

(2) Let $\{s_0, \ldots, s_m\}$ be the coefficient of the polynomial $\mathtt{P}_f$, let $d \geq \sum_{i=0}^{n} s_i$ (so $d \geq 1$) and let $k \geq 1$ be also greater or equal to the degree of $\mathtt{P}_f$. The thesis can be rewritten as $\mathtt{P}_f(a_1, \ldots, a_n) \leq (\sum_{i=1}^{m} s_i) \prod_{j=1}^{n} a_j^k = \sum_{i=1}^{m} (s_i \prod_{j=1}^{n} a_j^k)$. Moreover there exists

$$k_{1,1} \ldots, k_{1,n}, k_{2,1}, \ldots, k_{2,n}, \ldots, k_{m,1}, \ldots, k_{m,n}$$

such that $\mathtt{P}_f(a_1, \ldots, a_n) = \sum_{i=1}^{m} (s_i \prod_{j=1}^{n} a_j^{k_{i,j}})$ and for all $i \in [1, m]$ $k_{i,1} + \cdots + k_{i,n} \leq k$. Moreover $a_1, \ldots, a_n \in A \setminus \{0\}$, and therefore the thesis trivially holds since for all $i \in [1, m]$ $s_i \prod_{j=1}^{n} a_j^{k_{i,j}} \leq s_i \prod_{j=1}^{n} a_j^k$.

(3) By induction of t. If t is a variable, then $|t| = 1$ and $\pi_a(t) = a \leq 2^a \leq 2^{2^{\log_2(d)}} \leq 2^{2^{c|t|}}$. If t is of the form $f(t_1, \ldots, t_n)$ then $\pi_a(t) = \mathtt{P}_f(\pi_a(t_1), \ldots, \pi_a(t_n))$. By inductive hypothesis, since $\mathtt{P}_f$ is monotone, $\pi_a(t) \leq \mathtt{P}_f(2^{2^{c|t_1|}}, \ldots, 2^{2^{c|t_n|}})$. From (2) it follows that $\mathtt{P}_f(2^{2^{c|t_1|}}, \ldots, 2^{2^{c|t_n|}}) \leq d \prod_{i=1}^{n} (2^{2^{c|t_i|}})^k = d 2^{\Sigma_i (k 2^{c|t_i|})} = 2^{\log_2(d)} 2^{\Sigma_i (k 2^{c|t_i|})} = 2^{\log_2(d) + k \Sigma_i (2^{c|t_i|})} \leq 2^{(\log_2(d)+k) \Sigma_i (2^{c|t_i|})}$. Since $d \geq a \geq 1$ and $k \geq 1$ it holds that $c \geq 1$ and therefore $2^{(\log_2(d)+k) \Sigma_i (2^{c|t_i|})} \leq 2^{c \Pi_i (2^{c|t_i|})} \leq 2^{2^c \Pi_i (2^{c|t_i|})} \leq 2^{2^{c|t|}}$.

(4) W.l.o.g. consider $s = l\sigma$ and $t = r\sigma$ for a rewriting rule $l \to r$ and a valuation σ. The number of occurrences of f in $l\sigma$ is $|l|_f + \sum_{p \in \{p | l|_p \in X\}} |\sigma(l|_p)|_f$ where $|l|_f$ only depends on the left side of the rewriting rule. Similarly, $|r\sigma|_f = |r|_f + \sum_{p \in \{p | r|_p \in X\}} |\sigma(r|_p)|_f$ where $|r|_f$ depends only on the right side of the rewriting rule. Let V the number of variables in r (i.e. $|\{p | r|_p \in X\}|$). It holds that $|r\sigma|_f \leq |r|_f + V \max_{p \in \{p | r|_p \in X\}} (|\sigma(r|_p)|_f)$. Since every variable of r also occurs in l it must hold that $|r\sigma|_f \leq |r|_f + V \max_{p \in \{p | l|_p \in X\}} (|\sigma(l|_p)|_f)$. Moreover $\max_{p \in \{p | l|_p \in X\}} (|\sigma(l|_p)|_f)$ is trivially less or equal that all the occurrences of f in $l\sigma$, therefore $|r\sigma|_f \leq |r|_f + V |l\sigma|_f \leq (|r|_f + V)(|l\sigma|_f + 1)$. $|r|_f$ and V only depends on the rule itself. Let k be greater or equal than the maximum number of occurrences of f in the right side of each rule of the TRS plus the number of variables in the right side of each rule of the TRS. it holds that $|r\sigma|_f \leq k(|l\sigma|_f + 1)$.

(5) From the above point, it holds that for all terms s and t such that $s \to t$, $|t|_{\mathtt{s}} \leq k(|s|_{\mathtt{s}} + 1)$. So at each step of the rewriting system, the number of $\mathtt{s}$ can at most increase k times (from the previous proof, for Ackermann this should hold for $k \geq 5$). If Ackermann could be proved terminating using a polynomial interpretation over integers then given any term t, the maximum number of steps will be $\pi_a(t) \leq 2^{2^{c|t|}}$, where c is fixed (and depends on the polynomial interpretation, see proof (2)). The size of a term of the form $\mathtt{a}(m, n)$ is $m + n + 3$. We conclude that there must exists k and c such that for any X, Y it should hold $Ack(X, Y) \leq k * 2^{2^{c(X+Y+3)}}$. This cannot hold since $Ack(X, Y)$ is not primitive recursive whereas $k * 2^{2^{c(X+Y+3)}}$ is, and therefore there exists X and Y such that $Ack(X, Y) > k * 2^{2^{c(X+Y+3)}}$.