Rewriting Techniques: TD 6

21-12-2017

A pair $(A, \leq)$, where $\leq$ is a binary relation on the set A, is a **well quasi-ordering** (wqo) if $\leq$:

- is a *quasi-order*, i.e. $\leq$ is reflexive and transitive;
- is *well-founded*, i.e. there are no infinite strictly decreasing sequences $a_0 > a_1 > a_2 > \dots$ in A;
- does not have *infinite antichains*, i.e. it does not exists an infinite subset I of A such that for each $a, b \in I$, $a \not\leq b$ and $b \not\leq a$.

Exercise 1 :
Which of the following are wqo?

1. $(\mathbb{N}, \leq)$
2. $(\mathbb{Z}, \leq)$
3. $(\Sigma^*, \leq)$ where $\leq$ is the prefix order over a finite alphabet Σ
4. $(\mathbb{N}, |)$ where $n|m$ iff n divides m
5. $(\mathcal{P}(\mathbb{N}), \subseteq)$
6. $(\mathcal{P}(\mathbb{N}), \sqsubseteq)$ where $U \sqsubseteq V$ iff for all $m \in V$ there is $n \in U$ such that $n \leq m$
7. $(\Sigma^*, \leq_{\text{lex}^*})$ where $\leq$ is a strict order over a finite alphabet Σ and its extension $\leq_{\text{lex}^*}$ to Σ^* is defined as: $a_0 \dots a_n \leq_{\text{lex}^*} b_0 \dots b_m$ iff $a_0 \dots a_n$ is a prefix of $b_0 \dots b_m$ or there is $i \leq \min(n, m)$ such that $a_i < b_i$ and for all $0 \leq j < i$, $a_j = b_j$.
8. $(\{(a, b) \in \mathbb{N}^2 \mid a < b\}, \preceq)$ where $(a, b) \preceq (a', b')$ iff $(a = a' \wedge b \leq b')$ or $b < a'$.

Solution:

(1) Yes, it's total and well-founded.

(2) No, $(-n)_{n\in\mathbb{N}}$ is strictly decreasing.

(3) For $n = 1$, yes. For $n \geq 2$ no, $(a^n b)_{b\in\mathbb{N}}$ is an infinite antichain.

(4) No, the set of prime numbers is an infinite antichain.

(5) No, $(\{n\})_{n\in\mathbb{N}}$ is an infinite antichain.

(6) Yes, since $U \sqsubseteq V$ if and only if $V = \emptyset$ or U, V are not empty and $\min(U) \leq \min(V)$. So, two sets will be in the same equivalence class whenever they are both empty or they have the same minimal element. The domain of equivalence classes is isomorphic to $(\mathbb{N} \cup \{\top\}, \leq)$ where $\forall n \in \mathbb{N}\ n < \top$.

(7) For $n = 1$, yes. For $n \geq 2$, no, $(a^n b)_{n\in\mathbb{N}}$ is strictly decreasing.

(8) Yes. It's trivially reflexive, whereas transitivity can be shown by considering the following four cases. Let $(a, b) \preceq (a', b')$ and $(a', b') \preceq (a'', b'')$.

- if $a = a'$, $b \leq b'$, $a' = a''$ and $b' \leq b''$ then $a = a''$ and $b \leq b''$, therefore $(a, b) \preceq (a'', b'')$;
- if $a = a'$, $b \leq b'$ and $b' < a''$ then $b < a''$, therefore $(a, b) \preceq (a'', b'')$;
- if $b < a'$, $a' = a''$ and $b' \leq b''$ then $b < a''$, therefore $(a, b) \preceq (a'', b'')$;
- if $b < a'$, $b' < a''$ and $a' < b'$ by definition, then $b < a''$, therefore $(a, b) \preceq (a'', b'')$;

It follows that $\preceq$ is a quasi-order.

It is also well-founded because for every (a', b'), the set $\{(a, b) \mid (a, b) \preceq (a', b')\}$ is finite. To show this it is sufficient to write the set as the union $\{(a, b) \in \mathbb{N}^2 \mid a = a' \wedge b \leq b'\} \cup \{(a, b) \in \mathbb{N}^2 \mid a < b \wedge b < a'\}$ and prove the finiteness of its two sets.

Lastly, each antichain is finite. From the definition of $\preceq$, two elements (a, b) and (a', b') are incomparable whenever $a \neq a'$, $a \leq b'$ and $b \leq a'$. Let $((a_i, b_i))_{i \in I}$ be an antichain. It must hold that for every $i, j \in I$, $a_i = a_j$ implies $i = j$ (since $a \neq a'$). Moreover, for each $i \in I$, $a_i \leq b_1$. It follows that I is finite.

Let $(A, \leq)$ a quasi-ordering. We say that $a \in A$ is **minimal** whenever there are no $a' \in A$ such that $a' < a$. Given a subset U of A, we note $\uparrow U$ the **upward-closure** of U, i.e. $\{a \in A \mid \exists a' \in U,\ a' \leq a\}$.

Exercise 2 :
Let $(A, \leq)$ be a quasi-ordering.

1. Show that if $\leq$ is well-founded, then every element is larger then or equal to a minimal element.
2. Prove that $(A, \leq)$ is a wqo iff every non-empty subset of A has at least one minimal element and at most a finite number of minimal element up to equivalence.
3. Prove that if $(A, \leq)$ is a wqo then any of its upward-closed subsets can be written as $\uparrow\{a_1, \ldots, a_n\}$ for some $a_1, \ldots, a_n \in A$.
4. Prove that $(A, \leq)$ is a wqo if and only if any increasing sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_k \subseteq \ldots$ of upward-closed subsets of A stabilizes, i.e. there is $p \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $U_{p+i} = U_p$.

Solution:

(1) Let $a_0 \in A$. Assume that there is no a minimal such that $a \leq a_0$ (and therefore in particular a is not minimal). Since a_0 is not minimal, there exists $a_1 < a_0$ and there is no a minimal such that $a \leq a_1$. We can therefore reason by induction w.r.t. the hypothesis "there is no a minimal such that $a \leq a_i$". Assume that $a_i < a_{i-1} < a_{i-2} < \cdots < a_0$ are constructed. By hypothesis, a_i is not minimal, then there is $a_{i+1} < a_i$. By doing so we construct an infinite strictly decreasing sequence, which contradicts the well-foundness of A.

(2) As a wqo is well-founded, each subset of A is well-founded. If this subset is non-empty then it has at least a minimal element. Assume now that there is an infinite number of minimal elements up to equivalence and let $(a_i)_{i \in \mathbb{N}}$ be an infinite sequence of non-equivalent minimal elements. As they are minimal and not equal, they are incomparable and therefore $(a_i)_{i \in \mathbb{N}}$ is an infinite antichain.

For the converse,

- suppose A not well-founded. Then there exists a subset of A corresponding to an infinite strictly decreasing sequence. By definition this set is non-empty and without minimal elements. From the first hypothesis it follows that $\leq$ must be well-founded.
- Suppose now that there exists a subset of A corresponding to an infinite antichain. By definition the subset contains an infinite number of incomparable elements which are minimal in this subset. From the second hypothesis it follows that $\leq$ does not have infinite antichains.

We conclude that $(A, \leq)$ is a wqo.

(3) Let U be upward-closed. From the previous point, it has a finite number of minimal elements up to equivalence. Let $a_1, \ldots, a_n$ be elements representing those equivalence classes. We will prove that $U = \uparrow\{a_1, \ldots, a_n\}$. $U \supseteq \uparrow\{a_1, \ldots, a_n\}$ holds since all a_i, $i \in [1, n]$ belongs to U and U is upward-closed. To prove $U \subseteq \uparrow\{a_1, \ldots, a_n\}$ notice that U is well-founded since A is well-founded. From the first point of this exercise, every element of U is greater or equal to a minimal element. Since every minimal element is equivalent to an element in $\{a_1, \ldots, a_n\}$, we conclude that every element of U is in $\uparrow\{a_1, \ldots, a_n\}$.

(4) Suppose A wqo and assume that there is an increasing sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_k \subseteq \ldots$ that does not stabilize and let $U_i \subset U_{i+1}$ two elements of the sequence. From the previous point, there are $A_{U_i} = \{a_1, \ldots, a_n\}$ and $A_{U_{i+1}} = a'_1, \ldots, a'_m$ such that $U_i = \uparrow A_{U_i}$ and $U_{i+1} = \uparrow A_{U_{i+1}}$. Since $U_i \subset U_{i+1}$, $A_{U_i} \subset U_{i+1}$ and from the definition of upward-closed set it must hold that there is $a' \in A_{U_{i+1}}$ which is not in U_i and such that

- it is incomparable with every $a \in A_{U_i}$, or
- there is $a \in A_{U_i}$ $a' < a$.

If the sequence does not stabilize then, at each step, it is possible to find a new element of A with such properties. It follows that A must contain an infinite strictly decreasing sequence or an infinite antichain, which contradicts the hypothesis of A wqo.

For the converse, suppose that any increasing sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_k \subseteq \ldots$ of upward-closes subsets of A stabilizes and assume A not wqo. Since A is a quasi-ordering, one of the following must hold:

- there is an infinite strictly decreasing sequence $a_0 > a_1 > a_2 > \ldots$ in A. Then consider the increasing sequence $(\uparrow\{a_i\})_{i \in \mathbb{N}}$ of upward-closed sets. Since the sequence $a_0 > a_1 > a_2 > \ldots$ is not stabilizing, we have a contradiction and $\leq$ must be well-founded.
- there is an infinite subset I of A such that for each $a, b \in I$, $a \not\leq b$ and $b \not\leq a$. Consider the increasing sequence $U_0 = \emptyset$, $U_i = U_{i-1} \cup \uparrow\{a\}_{a \in I \setminus U_{i-1}}$ of upward-closed subsets of A. Since the sequence is not stabilizing, we have a contradiction and $\leq$ does not have infinite antichains.

Let $(A_1, \leq_1)$, $\ldots$, $(A_n, \leq_n)$ be non-empty quasi-orderings. Their **product extension** is the quasi ordering $(A_1 \times \cdots \times A_n, \leq_\times)$ with $(x_1, \ldots, x_n) \leq_\times (y_1, \ldots, y_n)$ whenever for each $i \in [1, n]$ $x_i \leq_i y_i$.

Exercise 3 :
Prove that $(A_1, \leq_1)$, $\ldots$, $(A_n, \leq_n)$ are wqo iff their product extension $(A_1 \times \cdots \times A_n, \leq_\times)$ is a wqo.

Solution:

Reflexivity and transitivity are preserved and therefore $(A_1 \times \cdots \times A_n, \leq_\times)$ is a quasi-order whenever the same holds for $(A_1, \leq_1)$, $\ldots$, $(A_n, \leq_n)$. For example $(a_1, \ldots, a_n) \leq_\times (a_1, \ldots, a_n)$ if and only if for all $i \in [1, n]$ $a_i \leq a_i$.

Also well-foundness is preserved. Suppose A_i not well-founded. Then there exists an infinite decreasing sequence $b_1 >_i b_2 >_i b_3 >_i \ldots$. Let $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ be elements of $A_1, \ldots, A_{i-1}, A_{i+1}, A_n$ respectively. Then

$$(a_1, \ldots, a_{i-1}, b_1, a_{i+1}, \ldots, a_n) <_\times (a_1, \ldots, a_{i-1}, b_2, a_{i+1}, \ldots, a_n) <_\times \cdots$$

is an infinite decreasing sequence and $(A_1 \times \cdots \times A_n, \leq_\times)$ is not well-founded. The converse follows with similar arguments: if $(A_1 \times \cdots \times A_n, \leq_\times)$ is not well-founded then an infinite decreasing sequence of its elements implies the existence of a decreasing sequence of elements in at least one domain A_i, $i \in [1, n]$.

Lastly, also the presence of antichains is preserved. For example, suppose that A_i contains an infinite antichain. Then there exists an infinite subset $\{b_1, b_2, b_3, \ldots\}$ of incomparable elements of A_i. Let $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ be elements of $A_1, \ldots, A_{i-1}, A_{i+1}, A_n$ respectively. Then

$$\{(a_1, \ldots, a_{i-1}, b_1, a_{i+1}, \ldots, a_n), (a_1, \ldots, a_{i-1}, b_2, a_{i+1}, \ldots, a_n), \ldots\}$$

is an infinite subset of incomparable elements of $(A_1 \times \cdots \times A_n, \leq_\times)$. The converse follows with similar arguments: $(A_1 \times \cdots \times A_n, \leq_\times)$ contains an infinite antichain then the same holds true for at least one domain A_i, $i \in [1, n]$.

Let $(A_1, \leq_1)$, ..., $(A_n, \leq_n)$ be non-empty quasi-orderings. Their **lexicographic extension** is the quasi ordering $(A_1 \times \cdots \times A_n, \leq_{\text{lex}})$ with $(x_1, \ldots, x_n) \leq_\times (y_1, \ldots, y_n)$ whenever there is $j \in [1, n+1]$ such that for each $i < j$ it holds $x_i =_i y_i$ and if $j \leq n$ then $x_j <_j y_j$.

Exercise 4 :
Prove that $(A_1, \leq_1)$, ..., $(A_n, \leq_n)$ are wqo iff their lexicographic extension $(A_1 \times \cdots \times A_n, \leq_{\text{lex}})$ is a wqo.

Solution:

Reflexivity and transitivity are preserved and therefore $(A_1 \times \cdots \times A_n, \leq_{\text{lex}})$ is a quasi-order whenever the same holds for $(A_1, \leq_1)$, ..., $(A_n, \leq_n)$. For example $(a_1, \ldots, a_n) \leq_{\text{lex}} (a_1, \ldots, a_n)$ if and only if for all $i \in [1, n]$ $a_i \leq a_i$.

Moreover, from the definition it is easy to see that $\leq_\times \subseteq \leq_{\text{lex}}$ and therefore, if $\leq_{\text{lex}}$ is well-founded then $\leq_\times$ is well-founded and from the previous exercise it holds that $(A_1, \leq_1)$, ..., $(A_n, \leq_n)$ are well-founded. Similarly, if $\leq_{\text{lex}}$ does not have infinite antichain then each $(A_i, \leq_i)$, $i \in [1, n]$ does not have any infinite antichain.

We will now prove that if $(A_1 \times \cdots \times A_n, \leq_{\text{lex}})$ has an infinite antichain then there exists $(A_i, \leq_i)$, $i \in [1, n]$, with an infinite antichain. Lets consider the simpler case $(A_1, \leq_1)$, $(A_2, \leq_2)$ and $(A_1 \times A_2, \leq_{\text{lex}})$. The proof can be easily adapted to the general case. Let I be an infinite antichain in $(A_1 \times A_2, \leq_{\text{lex}})$. As such, each pair of elements $(a, b), (a', b') \in I$ are such that $(a, b) \not\leq_{\text{lex}} (a', b')$ and $(a', b') \not\leq_{\text{lex}} (a, b)$. From the definition of $\leq_{\text{lex}}$ it therefore holds

$$(a \not\leq_1 a' \wedge a' \not\leq_1 a) \vee (a \not<_1 a' \wedge a' \not<_1 a \wedge (b \not\leq_2 b' \wedge b' \not\leq_2 b))$$

So it must hold that a and a' are incomparable or they are equal but b and b' are incomparable. If I has an infinite number of pairs such that their first component is incomparable, then $(A_1, \leq_1)$ contains an infinite antichain. Otherwise, suppose I has a finite number of pairs such that their component is incomparable. Then I contains an infinite number of pairs such that the first component is equal. Consider $I' \subseteq I$ the infinite set of such pairs. It holds that b and b' are incomparable for every two pairs (a, b) and (a, b') in I'. It follows that $(A_1, \leq_2)$ contains an infinite antichain. A similar proof can be made to show that if $(A_1 \times \cdots \times A_n, \leq_{\text{lex}})$ is not well-founded then there exists a $(A_i, \leq_i)$, $i \in [1, n]$, not well-founded.

A quasi-ordering $(A, \leq)$ is **total** if for every $x, y \in A$, $x \leq y$ or $y \leq x$.

A **linearization** of $(A, \leq)$, where $\leq$ is a binary relation on the set A, is a total quasi-ordering $(A, \sqsubseteq)$ such that for all $x, y \in A$, $x \leq y$ implies $x \sqsubseteq y$ and $x < y$ implies $x \sqsubset y$.

Exercise 5 :

1. Prove that a total quasi-ordering is a wqo if and only if it is well-founded.
2. Prove that every quasi-ordering has a linearization. *(hint: use Zorn's lemma)*
3. Prove that a quasi-ordering is a wqo if and only if all its linearization are well-founded.

Solution:

(1) A total quasi-ordering cannot have infinite antichains.

(2) Let $(A, \leq)$ be a quasi-ordering. Let $\Gamma = \{(A, \sqsubseteq) \mid \sqsubseteq \text{ is a quasi-order}, \leq \subseteq \sqsubseteq \text{ and } < \subseteq \sqsubset\}$. Γ is non-empty since $(A, \leq) \in \Gamma$. Consider the binary relation $\to$ s.t. $(A, \leq_1) \to (A, \leq_2)$ whenever $\leq_1 \subseteq \leq_2$. $\to$ is a partial order such that each of its chain has an upper-bound in Γ. By applying Zorn's lemma, Γ has at least one maximal element $(A, \preceq)$. We show that $(A, \preceq)$ is a linearization of $(A, \leq)$ by showing that is total. Suppose it is not. So there are x, y in A such that $x \not\preceq y$ and $y \not\preceq x$. Define $\preceq' = (\preceq \cup \{(x, y)\})^*$. Then $\preceq'$ is a quasi-order such that $\preceq \subseteq \preceq'$ and therefore $\leq \subseteq \preceq'$. To have a contradiction, it remains to prove that $\prec \subseteq \prec'$. Assume $\alpha \prec \beta$. It holds that $\alpha \preceq \beta$ and $\beta \not\preceq \alpha$. By definition, $\alpha \preceq' \beta$. Assume $\beta \preceq' \alpha$. This means

that $\beta \preceq x \preceq' y \preceq \alpha$. Thus, $y \preceq \alpha \preceq \beta \preceq x$, which is absurd since $x \not\preceq y$ and $y \not\preceq x$. It follows that $\preceq$ must be total.

(3) Let $(A, \leq)$ be a quasi-ordering. Let $\preceq$ be a linearization of $\leq$. Assume that it exists an infinite sequence S: $x_1 \succ x_2 \succ \dots$. Suppose that S contains an infinite set of incomparable elements w.r.t. $\leq$. Then it's an infinite antichain and $\leq$ is not a wqo. If, instead, the set of incomparable elements is finite, then there must be an infinite subsequence of S ordered by $>$ and therefore $\leq$ cannot be a wqo since it's not well-founded. For the converse, if every linearization is well-founded, then $\leq$ is also well-founded since a strictly decreasing sequence for $\leq$ is also a strictly decreasing sequence for any linearization. Assume now that $\leq$ has an infinite antichain $(x_i)_{i \in \mathbb{N}}$. Let $\leq' = (\leq \cup \{(x_j, x_i) \mid i < j\})^*$. This is a quasi-ordering which satisfies

- $\leq \subseteq \leq'$;
- $< \subseteq <'$: if $x \leq y$ and $y \not\leq x$ then $x \leq' y$. Assume $y \leq' x$ then there are $i < j$ such that $y \leq x_j$ and $x_i \leq x$. So $x_i \leq x_j$ which is absurd.

It follows that any linearization of $\leq'$ is also a linearization of $\leq$. But $\leq'$ is not well-founded, and therefore $\leq$ does not contain any infinite antichain.

A **topology** on a set X is a set $\mathcal{O}(X)$ of (open) subsets of $\mathcal{P}(X)$ that is closed by unions and finite intersections and contains both $\emptyset$ and X. A **topological space** $(X, \mathcal{O}(X))$ is a set with a topology on it.

Let $(X, \mathcal{O}(X))$ be a topological space. A subset K of X is said to be **compact** if for every subset $\mathcal{K} \subseteq \mathcal{O}(X)$ such that $K \subseteq \bigcup \mathcal{K}$ there is a finite subset $\mathcal{F}(\mathcal{K}) \subseteq \mathcal{K}$ such that $K \subseteq \bigcup \mathcal{F}(\mathcal{K})$. We say that a topological space $(X, \mathcal{O}(X))$ is **Noetherian** if and only if every $K \in \mathcal{O}(X)$ is compact.

Exercise 6 :
Let $(A, \leq)$ be a quasi-ordering and let $\widehat{\mathcal{O}(A)}$ be the set of all upward-closed subsets of A (this is known as the Alexandrov's topology on quasi-ordering).

1. Prove that $\widehat{\mathcal{O}(A)}$ is a topology.
2. Prove that $(A, \leq)$ is a wqo if and only if $\widehat{\mathcal{O}(A)}$ is Noetherian.

Solution:

(1) The intersection and the union of upward-closed sets is again an upward-closed set. Moreover $\widehat{\mathcal{O}(A)}$ contains both $\emptyset$ and A. We conclude that it is a topology.

(2) Suppose $(A, \leq)$ wqo. Let U be an upward-closed set of A and $\mathcal{K}$ be a set of upward-closed sets such that $U \subseteq \bigcup \mathcal{K}$. From the third point of Exercise 2, there are $a_1, \dots, a_n$ such that $U = \uparrow\{a_1, \dots, a_n\}$. It holds that for each $i \in [1, n]$ there is $U_i \in \mathcal{K}$ such that $a_i \in U_i$. As the U_i are upward-closed sets, $U \subseteq \bigcup_{i \in [1,n]} U_i$. For the converse, we prove that if $\widehat{\mathcal{O}(A)}$ is Noetherian then any increasing sequence $U_1 \subseteq U_2 \subseteq U_3 \dots$ of upward-closed subsets of A stabilize. Then, the result follows from the fourth point of Exercise 2. Let $U_1 \subseteq U_2 \subseteq U_3 \dots$ be an increasing sequence of sets in $\widehat{\mathcal{O}(A)}$. Let $U = \bigcup_i U_i$, which is an upward-closed set. By hypothesis there are $i_1 < \dots < i_k$ such that $U = \bigcup_{i \in \{i_1, \dots, i_k\}} U_{i_k}$. Then for all $i \geq i_k$, $U = U_i$ and the sequence stabilize.

A **multi-context** C is a term with distinguished variables $\square_1, \dots, \square_n$ occurring exactly once. Replacing them by terms $t_1, \dots, t_n$ respectively is denoted by $C[t_1, \dots, t_n]$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two disjoint signatures. A symbol is of **color** $k \in \{1, 2\}$ if it belongs to $\mathcal{F}_k$. A term t is of color k if it's not a variable and every symbol in it is of color k. We denote with $\overline{k}$ the other possible color of k, i.e. $3 - k$.

Let t be a term with symbols in $\mathcal{F}_1 \cup \mathcal{F}_2$. We define cap($t$) and aliens($t$) respectively as

$$\mathrm{cap}(t) = \begin{cases} x & \text{if } t = x \text{ is a variable} \\ C & \text{if } t = C[t_1, \dots, t_n] \text{ where } C \text{ is of color } k \in \{1,2\} \text{ and} \\ & t_1, \dots, t_n \text{ are headed by symbols of color } \overline{k} \end{cases}$$

$$\mathrm{aliens}(t) = \begin{cases} \emptyset & \text{if } t \text{ is a variable} \\ \{\!|t_1, \dots, t_n|\!\} & \text{if } t = C[t_1, \dots, t_n] \text{ where } C \text{ is of color } k \in \{1,2\} \text{ and} \\ & t_1, \dots, t_n \text{ are headed by symbols of color } \overline{k} \end{cases}$$

The **rank** of a term t, denoted with $\mathrm{rk}(t)$, is the maximum number of color layers in t, i.e. $\mathrm{rk}(t) = 1 + \max_{a \in \mathrm{aliens}(t)}(\mathrm{rk}(a))$.

Exercise 7 :
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two disjoint signatures and let $\mathcal{R}_1$, $\mathcal{R}_2$ be two TRSs on $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively such that $\to_{\mathcal{R}_1}$ terminates on $T_1 = T(\mathcal{F}_1, V)$ and $\to_{\mathcal{R}_2}$ terminates on $T_2 = T(\mathcal{F}_2, V)$, where V is a set of variables. Let $\to$ be the rewrite relation on $T = T(\mathcal{F}_1 \cup \mathcal{F}_2, V)$ generated by $\mathcal{R}_1 \cup \mathcal{R}_2$.

1. Prove that for each term $t, u \in T$, if $t \to u$ then $\mathrm{rk}(t) \geq rk(u)$.
2. Prove that if $\mathcal{R}_1$ and $\mathcal{R}_2$ do not have any rules of the form $l \to x$ where x is a variable, then $\to$ terminates.
3. Prove that $\mathcal{R}_1 = \{\mathtt{a}(\mathtt{0}, \mathtt{1}, x) \to \mathtt{a}(x, x, x)\}$ and $\mathcal{R}_2 = \{\mathtt{m}(x, y) \to x,\ \mathtt{m}(x, y) \to y\}$ are terminating, whereas $\mathcal{R}_1 \cup \mathcal{R}_2$ is not.

Solution:
(1) The proof is by induction on $\mathrm{rk}(t)$. We need to distinguish two cases:

- The reduction that leads to u is in $\mathrm{cap}(t)$. If u is a variable, then $\mathrm{rk}(u) = 1 \leq \mathrm{rk}(t)$. If $\mathrm{cap}(t)$ and $\mathrm{cap}(u)$ have distinct colors, then u is an alien of t and $\mathrm{rk}(u) < \mathrm{rk}(t)$. Otherwise it must holds that $\mathrm{cap}(t) \to_{\mathcal{R}_1} \mathrm{cap}(u)$ or $\mathrm{cap}(t) \to_{\mathcal{R}_2} \mathrm{cap}(u)$ (if $\mathrm{cap}(t)$ is of color 1 or 2 respectively) and $\mathrm{aliens}(u) \subseteq \mathrm{aliens}(t)$. Therefore, $\mathrm{rk}(u) \leq \mathrm{rk}(t)$.
- Let $t = C[t_1, \dots, t_n]$ where $C = \mathrm{cap} t$. Suppose now that reduction is in $t_i \in \mathrm{aliens} t$, $i \in [1, n]$, and it reduces t_i to t_i' (therefore, $u = C[t_1, \dots, t_{i-1}, t_i', t_{i+1}, \dots, t_n]$). By induction hypothesis (since $\mathrm{rk}(t_i) < \mathrm{rk}(t)$), $\mathrm{rk}(t_i') \leq \mathrm{rk}(t_i)$. By definition of rk it holds that $\mathrm{rk}(t) = \mathrm{rk}(C[t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n]) \geq \mathrm{rk}(C[t_1, \dots, t_{i-1}, t_i', t_{i+1}, \dots, t_n]) = \mathrm{rk}(u)$.

(2) Let t be a term. The proof is by well-founded induction on $(\mathrm{rk}(t), \mathrm{cap}(t), \mathrm{aliens}(t))$ w.r.t. the well-founded order $(>, (\to_{\mathcal{R}_1}^+ \cup \to_{\mathcal{R}_2}^+), \to_{\mathrm{mul}})_{\mathrm{lex}}$. Notice that, since $\to_{\mathcal{R}_1}$ and $\to_{\mathcal{R}_2}$ are terminating, the relation $(\to_{\mathcal{R}_1}^+ \cup \to_{\mathcal{R}_2}^+) \subseteq (T(\mathcal{F}_1, V) \cup T(\mathcal{F}_2, V))^2$ that maps elements of $T(\mathcal{F}_1, V)$ to elements of $T(\mathcal{F}_1, V)$ and elements of $T(\mathcal{F}_2, V)$ to elements of $T(\mathcal{F}_2, V)$ is well-founded. $\to_{\mathrm{mul}}$ is also well-founded as explained in the last point of the proof. If t is irreducible, then $\to$ terminates on it. Instead, if $t \to u$, then we need to consider the following three cases:

- $\mathrm{rk}(t) = 1$, i.e. t is of color $k \in \{1, 2\}$. Then u is of the same color and therefore $t \to_{\mathcal{R}_k} u$. We can apply the induction hypothesis (u terminates) and conclude that also t terminates.
- $\mathrm{rk}(t) > 1$ and the reduction is in $C = \mathrm{cap}(t)$. Let k be the color of C. Since the rules are non-collapsing, then $\mathrm{cap}(u)$ is of color k and, by well-foundness of $\to_{\mathcal{R}_k}$ we can apply the induction hypothesis (u terminates) and conclude that, t also terminates.
- Lastly, $\mathrm{rk}(t) > 1$ and the reduction is in some alien $a \in \mathrm{aliens}(t)$, that is reduced to a'. Then, since $\mathrm{rk}(a) < \mathrm{rk}(t)$, we can apply the induction hypothesis and conclude that a is terminating. Since the rules are non-collapsing, it holds that $\mathrm{cap}(t) = \mathrm{cap}(u)$ and, from $\mathrm{aliens}(u) = \mathrm{aliens}(t) - \{\!|a|\!\} + \{\!|a'|\!\}$, all aliens of t and u are terminating. Therefore, it holds that $\to_{\mathrm{mul}}$ is a well-founded strict order when restricted to elements of $\mathrm{aliens}(t)$. Since $\mathrm{aliens}(t) \to_{\mathrm{mul}} \mathrm{aliens}(u)$, u terminates and so does t.

(3) For $\mathcal{R}_1$ consider the order $>$ where $t > s$ if and only if

$$\{\!|\ |u| \mid t = u \text{ or } \exists p \in \text{Pos}(t)\ t|_p = f(\mathtt{0}, \mathtt{1}, u) |\!\} >_{\text{mul}} \{\!|\ |u| \mid s = u \text{ or } \exists p \in \text{Pos}(s)\ s|_p = f(\mathtt{0}, \mathtt{1}, u) |\!\}$$

It's easy to show that $>$ is a simplification order such that for each terms $t \to s$ it holds $t > s$. Instead, the termination of $\mathcal{R}_2$ is trivial (consider the size of the term). Lastly, $\mathcal{R}_1 \cup \mathcal{R}_2$ does not terminate:

$$\mathtt{a}(\mathtt{m}(\mathtt{0}, \mathtt{1}), \mathtt{m}(\mathtt{0}, \mathtt{1}), \mathtt{m}(\mathtt{0}, \mathtt{1})) \to \mathtt{a}(\mathtt{0}, \mathtt{m}(\mathtt{0}, \mathtt{1}), \mathtt{m}(\mathtt{0}, \mathtt{1})) \to \mathtt{a}(\mathtt{0}, \mathtt{1}, \mathtt{m}(\mathtt{0}, \mathtt{1})) \to \mathtt{a}(\mathtt{m}(\mathtt{0}, \mathtt{1}), \mathtt{m}(\mathtt{0}, \mathtt{1}), \mathtt{m}(\mathtt{0}, \mathtt{1}))$$