

Rewriting Techniques: TD 5

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A term is **linear** if no variable appears more than once in it. A TRS \mathcal{R} is called **left-linear** if for each of its rules $(l, r) \in \mathcal{R}$, l is linear. Similarly, \mathcal{R} is **right-linear** if for each of its rules $(l, r) \in \mathcal{R}$, r is linear.

We say that a TRS is **orthogonal** if it is left-linear and has no (non-trivial) critical pairs.

Theorem: Every orthogonal TRS is confluent.

In the following, given a TRS \mathcal{R} we will denote with $\rightarrow_{\mathcal{R}}$ (or simply \rightarrow when clear from the context) the rewrite relation built from it.

Exercise 1 :

Consider the TRS \mathcal{L}

$$\begin{aligned} @(\mathbf{a}, x) &\rightarrow x \\ @(@(\mathbf{s}, x), y) &\rightarrow x \\ @(@(@(\mathbf{p}, x), y), z) &\rightarrow @(@(x, z), @(y, z)) \end{aligned}$$

1. Is \rightarrow locally confluent?
2. Define a term Ω such that for all terms t , $@(\Omega, t) \rightarrow^+ @(t, t)$. Deduce that \rightarrow does not terminate (and therefore we cannot apply Newman's Lemma to prove confluency).
3. Is \mathcal{L} orthogonal? Is \rightarrow confluent?

Solution:

(1) A TRS is locally confluent if and only if all its critical pairs are joinable. Since \mathcal{L} does not have any non-trivial critical pair, it is locally confluent.

(2) The last rule is the only one that is not right-linear and therefore can be used to make copies of a subterm. Moreover we notice that $@(@(\mathbf{a}, z), @(\mathbf{a}, z)) \rightarrow^* @(z, z)$ using the first rule. Define $\Omega = @(@(\mathbf{p}, \mathbf{a}), \mathbf{a})$. For all terms t it holds

$$@(@(@(\mathbf{p}, \mathbf{a}), \mathbf{a}), t) \rightarrow @(@(\mathbf{a}, t), @(\mathbf{a}, t)) \rightarrow @(t, @(\mathbf{a}, t)) \rightarrow @(t, t)$$

We conclude that \mathcal{L} does not terminate since

$$@(@(@(\mathbf{p}, \mathbf{a}), \mathbf{a}), @(@(\mathbf{p}, \mathbf{a}), \mathbf{a})) \rightarrow^+ @(@(@(\mathbf{p}, \mathbf{a}), \mathbf{a}), @(@(\mathbf{p}, \mathbf{a}), \mathbf{a})).$$

(3) Since any orthogonal TRS is confluent (see Lecture) and \mathcal{L} is orthogonal, \mathcal{L} is confluent.

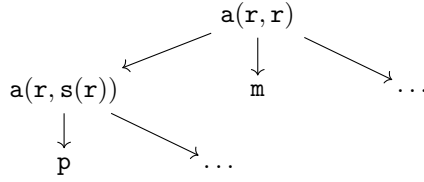
Exercise 2 :

Are the rewrite relations of the following TRSs confluent?

- $\mathbf{a}(x, x) \rightarrow \mathbf{m}$, $\mathbf{r} \rightarrow \mathbf{s}(\mathbf{r})$, $\mathbf{a}(x, \mathbf{s}(x)) \rightarrow \mathbf{p}$;
- $\mathbf{a}(x, x) \rightarrow \mathbf{m}$, $\mathbf{r} \rightarrow \mathbf{s}(\mathbf{r})$, $\mathbf{s}(x) \rightarrow \mathbf{a}(x, \mathbf{s}(x))$

Solution:

(1) The TRS is locally confluent (no non-trivial critical pairs), yet it is not confluent. For example the term $\mathbf{a}(\mathbf{r}, \mathbf{r})$ has two distinct normal forms \mathbf{m} and \mathbf{p} .



(2) First, we notice that in 4 steps we can reduce r to m :

$$r \rightarrow s(r) \rightarrow a(r, s(r)) \rightarrow a(s(r), s(r)) \rightarrow m$$

So now we know that we can replace r with m in any term. Let's therefore consider the term $a(r, s(r))$, i.e. the second element in the chain of reductions shown above. This element can be reduced to $a(m, s(m))$. This term cannot be reduced to m since the only reduction path is the infinite sequence

$$a(m, s(m)) \rightarrow a(m, a(m, s(m))) \rightarrow a(m, a(m, a(m, s(m)))) \rightarrow \dots$$

We conclude that the TRS is not confluent since from r i can reach both $a(m, s(m))$ and m .

A labelled rewrite relation $(\rightarrow, I, >)$ is a rewrite relation \rightarrow where arrows are equipped with labels from a set I and $>$ is a well-founded relation on I . If the arrow in $s \rightarrow t$ is labelled by ℓ , then we say that $s \xrightarrow{\ell} t$.

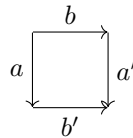
Let $(\rightarrow, I, >)$ be a labelled rewrite relation. The pair (a, b) representing the diagram $\xleftarrow{a} \xrightarrow{b}$ is called **local peak** (here, terms are omitted). Moreover, (a, b) is called (local) **critical peak** whenever it represent $t \xleftarrow{a} \xrightarrow{b} t'$ and (t, t') is a critical pair.

Given a strict order $>$ on a set I , the **lexicographic maximum order** $>_{\text{lmo}}$ is the relation on I^* such that $a >_{\text{lmo}} b$ if and only if $|a| >_{\text{mul}} |b|$ where $>_{\text{mul}}$ is the multiset order and $|\cdot|$ is defined recursively as follows:

- $|\epsilon| = \emptyset$
- $|ia| = \{i\} + (|a|/\{i\})$

where M/S is the multiset obtained from M by removing every occurrence of every element smaller than an element of S . $>_{\text{lmo}}$ is well-founded if the same holds true for $>$. We extend (\cdot/\cdot) to $(I^*)^2$ and we write a/b for $|a|/|b|$.

Let $(\rightarrow, I, >)$ be a labelled rewrite relation. A tuple (a, b, b', a') representing the diagram



where $a, a', b, b' \in I^*$, is called **decreasing diagram** if $b \geq_{\text{lmo}} b'/a$ and $a \geq_{\text{lmo}} a'/b$.

Theorem: A labelled rewrite relation $(\rightarrow, I, >)$ is confluent if for all of its local peaks (a, b) have a decreasing diagram (a, b, b', a') . A rewrite relation \rightarrow is confluent if there exists $(I, >)$ such that $(\rightarrow, I, >)$ is a confluent labelled rewrite relation.

Exercise 3 :

Prove Newman's lemma using decreasing diagrams techniques.

Solution:

Suppose \rightarrow be a terminating and locally confluent rewrite relation on terms T . Since \rightarrow is terminating, \rightarrow^+ is a well-founded order. We consider the labelled rewrite relation $(\rightarrow, T, \rightarrow^+)$ where each arrow $s \rightarrow t$ is labelled by s . As such, every local peak will have the form (s, s) for some $s \in T$. To prove the Lemma we just need to show that for every of such peaks there exists a decreasing diagram. Let (s, s) be the local peak corresponding to $s \rightarrow t'$ and $s \rightarrow t''$. Since \rightarrow is locally confluent, it holds that there exists a term t such that $t' \rightarrow^* t$ and $t'' \rightarrow^* t$. Let b' and a' be the concatenation of labels corresponding to $t' \rightarrow^* t$ and $t'' \rightarrow^* t$ respectively. It holds that $s \rightarrow^+ \bar{t}$ for each term \bar{t} in the path $t' \rightarrow^* t$ and in the path $t'' \rightarrow^* t$. As such, $\{s\}(\rightarrow^*)_{\text{mul}}|b'|/\{s\} = \emptyset$ and $\{s\}(\rightarrow^*)_{\text{mul}}|a'|/\{s\} = \emptyset$. We conclude that there exists a decreasing diagram from each local peak of $(\rightarrow, T, \rightarrow^+)$. By applying the Theorem above, \rightarrow is therefore confluent.

Exercise 4 :

1. Let R be a left and right linear TRS. For every, $(l, r) \in \mathcal{R}$, define $t \xrightarrow{(l,r)} s$ iff $t \rightarrow_{\mathcal{R}} s$ using the rule (l, r) . Assume given a well founded order $>$ on \mathcal{R} . Prove that if every critical peak of $(\rightarrow_{\mathcal{R}}, \mathcal{R}, >)$ has a decreasing diagram, then $\rightarrow_{\mathcal{R}}$ is confluent.

This principle is called the **rule-labelling heuristic**.

2. Consider the following TRS:

$$\begin{aligned} \text{nat} &\rightarrow 0 : \text{inc}(\text{nat}) \\ \text{inc}(x : y) &\rightarrow s(x) : \text{inc}(y) \\ \text{tl}(x : y) &\rightarrow y \\ \text{inc}(\text{tl}(\text{nat})) &\rightarrow \text{tl}(\text{inc}(\text{nat})) \end{aligned}$$

Prove its confluence using the rule-labelling heuristic.

3. Consider the following (non-confluent) TRS \mathcal{R} :

$$\begin{aligned} \mathbf{a} &\rightarrow \mathbf{b} \\ \mathbf{b} &\rightarrow \mathbf{a} \\ \mathbf{a} &\rightarrow 0 \\ \mathbf{b} &\rightarrow 1 \end{aligned}$$

Show that the rule-labelling heuristic cannot hold by proving that for any well-founded order $>$ on \mathcal{R} there exists a critical peak with no decreasing diagrams.

4. Consider the following TRS \mathcal{R} :

$$\begin{aligned} \mathbf{a} &\rightarrow \mathbf{b} \\ \mathbf{p}(\mathbf{a}) &\rightarrow \mathbf{s}(\mathbf{p}(\mathbf{a})) \\ \mathbf{p}(\mathbf{b}) &\rightarrow \mathbf{r} \\ \mathbf{s}(x) &\rightarrow \mathbf{m}(x, x) \\ \mathbf{m}(x, y) &\rightarrow \mathbf{r} \end{aligned}$$

Why the rule-labelling heuristic cannot be applied? Is it possible to find a set of labels I and a well-founded order $>$ on it such that every local peak of $(\rightarrow_{\mathcal{R}}, I, >)$ has a decreasing diagram?

Solution:

(1) If \mathcal{R} is left and right linear, then for each local peak (a, b) that is not a critical peak will hold that one of (a, b, b, ϵ) , (a, b, b, a) or (a, b, ϵ, a) is a decreasing diagram. Indeed, all three tuples satisfy $|b| \geq_{\text{mul}} |b'|/|a|$ and $|a| \geq_{\text{mul}} |a'|/|b|$, so it's only sufficient to show that the labelled rewrite relation $(\rightarrow_{\mathcal{R}}, \mathcal{R}, >)$ leads to one of those diagrams. This can be easily shown by considering the positions where the rewrite rules are applied. For example, consider $s \xrightarrow{(l,r)} t$ and $s \xrightarrow{(l',r')} t$ where (l, r) and (l', r') are applied in incomparable positions, i.e. there

exists two positions p, p' and two substitutions σ, σ' such that $p \not\leq p', p' \not\leq p, s|_p = l\sigma, s|_{p'} = l'\sigma'$ and $t = s[r\sigma]_p$ and $t' = s[r'\sigma']_{p'}$. Then from $p \not\leq p'$ and $p' \not\leq p$ it holds that $t[r'\sigma']_{p'} = t'[r\sigma]_p = s[r\sigma]_p[r'\sigma']_{p'}$. Therefore $((l, r), (l', r'), (l', r'), (l, r))$ is a decreasing diagram for $((l, r), (l', r'))$. A similar analysis can be done for non-critical local peak resulting from two rewrite rules applied in two positions p, p' where $p < p'$ (in this case is important to use the hypothesis of left and right linearity).

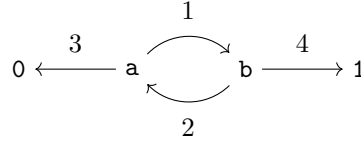
Since any non-critical local peak (a, b) has a decreasing diagram, if also every critical peak of $(\rightarrow_{\mathcal{R}}, \mathcal{R}, >)$ has a decreasing diagram, then by Theorem it follows that $\rightarrow_{\mathcal{R}}$ is confluent.

(2) The TRS is left and right linear. We refer to its rules as follows

- 1 : $\mathbf{nat} \rightarrow 0 : \mathbf{inc}(\mathbf{nat})$
- 2 : $\mathbf{inc}(x : y) \rightarrow s(x) : \mathbf{inc}(y)$
- 3 : $\mathbf{tl}(x : y) \rightarrow y$
- 4 : $\mathbf{inc}(\mathbf{tl}(\mathbf{nat})) \rightarrow \mathbf{tl}(\mathbf{inc}(\mathbf{nat}))$

The only critical pair arises from the term $\mathbf{tl}(\mathbf{inc}(\mathbf{nat}))$ and leads to the critical peak $(4, 1)$. The diagram $(4, 1, 123, 3)$ is decreasing by considering any ordering where 4 is greater than 1, 2 and 3. For example lets consider $4 > 3 > 2 > 1$. It holds that $\{3\} = |3| \geq_{\text{mul}} |123|/|4| = \emptyset$ and $\{4\} = |4| \geq_{\text{mul}} |3|/|1| = \{3\}$.

(3) We will refer to the rules of the TRS as: 1 ($\mathbf{a} \rightarrow \mathbf{b}$), 2 ($\mathbf{b} \rightarrow \mathbf{a}$), 3 ($\mathbf{a} \rightarrow 0$), 4 ($\mathbf{b} \rightarrow 1$). The TRS is therefore represented by the following diagram.



The TRS has two critical peaks $(1, 3)$ and $(2, 4)$. Let's consider the first one: after doing a 1 transition we can reach the irreducible state 0 only with paths of the form $2(12)^n 3$. We conclude that the diagrams for the critical peak $(1, 3)$ can be characterized by $\{(1, 3, 2(12)^n 3, \epsilon) \mid n \in \mathbb{N}\}$. It must therefore hold that $1 > 2$, otherwise for all n it will not hold that $|3| \geq_{\text{mul}} |2(12)^n 3|/|1|$. Similarly, by considering the second critical peak, we conclude that after doing a transition 2 we can reach the irreducible state 1 only with paths of the form $1(21)^n 4$. Therefore, the diagrams of the critical peak $(2, 4)$ can be characterized by $\{(2, 4, 1(21)^n 4, \epsilon) \mid n \in \mathbb{N}\}$. As such, it must hold that $2 > 1$, otherwise for all $n \in \mathbb{N}$ it will not hold that $|4| \geq_{\text{mul}} |1(21)^n 4|/|2|$. We conclude that it does not exist a well founded order $>$ on the TRS such that every of its critical peaks have a decreasing diagram for it.

(4) The rule-labelling heuristic requires the TRS to be right-linear, which is not the case here. In particular, the rule $\mathbf{s}(x) \rightarrow \mathbf{m}(x, x)$ can "self-duplicate". For example consider the term $\mathbf{s}(\mathbf{s}(r))$. Still, it is easy to find a decreasing labelling noting that the duplicated variable has on the right-hand side less \mathbf{s} symbols above it than on its left-hand side. Therefore, by labelling the steps first by the number of \mathbf{s} symbols above the term and then by the rule, we can prove that any local peak has a decreasing diagram by considering the lexicographic order on these new labels.