Rewriting Techniques: TD 5

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A term is **linear** if no variable appears more than once in it. A TRS $\mathcal{R}$ is called **left-linear** if for each of its rules $(l, r) \in \mathcal{R}$, l is linear. Similarly, $\mathcal{R}$ is **right-linear** if for each of its rules $(l, r) \in \mathcal{R}$, r is linear.

We say that a TRS is **orthogonal** if it is left-linear and has no (non-trivial) critical pairs.

Theorem: Every orthogonal TRS is confluent.

In the following, given a TRS $\mathcal{R}$ we will denote with $\to_{\mathcal{R}}$ (or simply $\to$ when clear form the context) the rewrite relation built form it.

Exercise 1 :
Consider the TRS $\mathcal{L}$

$$
\begin{aligned}
&@(\mathtt{a}, x) \to x\\
&@(@(\mathtt{s}, x), y) \to x\\
&@(@(@(\mathtt{p}, x), y), z) \to @(@(x, z), @(y, z))
\end{aligned}
$$

1. Is $\to$ locally confluent?
2. Define a term Ω such that for all terms t, $@(\Omega, t) \to^+ @(t, t)$. Deduce that $\to$ does not terminate *(and therefore we cannot apply Newman's Lemma to prove confluency)*.
3. Is $\mathcal{L}$ orthogonal? Is $\to$ confluent?

Solution:

(1) A TRS is locally confluent if and only if all its critical pairs are joinable. Since $\mathcal{L}$ does not have any non-trivial critical pair, it is locally confluent.

(2) The last rule is the only one that is not right-linear and therefore can be used to make copies of a subterm. Moreover we notice that $@(@(\mathtt{a}, z), @(\mathtt{a}, z)) \to^* @(z, z)$ using the first rule. Define $\Omega = @(@(\mathtt{p}, \mathtt{a}), \mathtt{a})$. For all terms t it holds

$$@(@(@(\mathtt{p}, \mathtt{a}), \mathtt{a}), t) \to @(@(\mathtt{a}, t), @(\mathtt{a}, t)) \to @(t, @(\mathtt{a}, t)) \to @(t, t)$$

We conclude that $\mathcal{L}$ does not terminate since

$$@(@(@(\mathtt{p}, \mathtt{a}), \mathtt{a}), @(@(\mathtt{p}, \mathtt{a}), \mathtt{a})) \to^+ @(@(@(\mathtt{p}, \mathtt{a}), \mathtt{a}), @(@(\mathtt{p}, \mathtt{a}), \mathtt{a})).$$

(3) Since any orthogonal TRS is confluent (see Lecture) and $\mathcal{L}$ is orthogonal, $\mathcal{L}$ is confluent.

Exercise 2 :
Are the rewrite relations of the following TRSs confluent?

- $\mathtt{a}(x, x) \to \mathtt{m},\ \mathtt{r} \to \mathtt{s}(\mathtt{r}),\ \mathtt{a}(x, \mathtt{s}(x)) \to \mathtt{p}$;
- $\mathtt{a}(x, x) \to \mathtt{m},\ \mathtt{r} \to \mathtt{s}(\mathtt{r}),\ \mathtt{s}(x) \to \mathtt{a}(x, \mathtt{s}(x))$

Solution:

(1) The TRS is locally confluent (no non-trivial critical pairs), yet it is not confluent. For example the term $\mathtt{a}(\mathtt{r}, \mathtt{r})$ has two distinct normal forms $\mathtt{m}$ and $\mathtt{p}$.

$$\begin{tikzcd}
 & & \mathtt{a(r,r)} \arrow[dll] \arrow[d] \arrow[dr] & \\
\mathtt{a(r,s(r))} \arrow[d] \arrow[dr] & & \mathtt{m} & \dots \\
\mathtt{p} & \dots & &
\end{tikzcd}$$

(2) First, we notice that in 4 steps we can reduce $\mathtt{r}$ to $\mathtt{m}$:

$$\mathtt{r} \;\to\; \mathtt{s(r)} \;\to\; \mathtt{a(r,s(r))} \;\to\; \mathtt{a(s(r),s(r))} \;\to\; \mathtt{m}$$

So now we know that we can replace $\mathtt{r}$ with $\mathtt{m}$ in any term. Let's therefore consider the term $\mathtt{a(r,s(r))}$, i.e. the second element in the chain of reductions shown above. This element can be reduced to $\mathtt{a(m,s(m))}$. This term cannot be reduced to $\mathtt{m}$ since the only reduction path is the infinite sequence

$$\mathtt{a(m,s(m))} \to \mathtt{a(m,\underline{a(m,s(m))})} \to \mathtt{a(m,a(m,\underline{a(m,s(m))}))} \to \dots$$

We conclude that the TRS is not confluent since from $\mathtt{r}$ i can reach both $\mathtt{a(m,s(m))}$ and $\mathtt{m}$.

A labelled rewrite relation $(\to, I, >)$ is a rewrite relation $\to$ where arrows are equipped with labels from a set I and $>$ is a well-founded relation on I. If the arrow in $s \to t$ is labelled by ℓ, then we say that $s \xrightarrow{\ell} t$.

Let $(\to, I, >)$ be a labelled rewrite relation. The pair (a, b) representing the diagram $\xleftarrow{a}\xrightarrow{b}$ is called **local peak** (here, terms are omitted). Moreover, (a, b) is called (local) **critical peak** whenever it represent $t \xleftarrow{a}\xrightarrow{b} t'$ and (t, t') is a critical pair.

Given a strict order $>$ on a set I, the **lexicographic maximum order** $>_{\mathrm{lmo}}$ is the relation on I^* such that $a >_{\mathrm{lmo}} b$ if and only if $|a| >_{\mathrm{mul}} |b|$ where $>_{\mathrm{mul}}$ is the multiset order and $|.|$ is defined recursively as follows:

- $|\epsilon| = \emptyset$
- $|ia| = \{\!|i|\!\} + (|a|/\{\!|i|\!\})$

where M/S is the multiset obtained from M by removing every occurrence of every element smaller than an element of S. $>_{\mathrm{lmo}}$ is well-founded if the same holds true for $>$. We extend $(./.)$ to $(I^*)^2$ and we write a/b for $|a|/|b|$.

Let $(\to, I, >)$ be a labelled rewrite relation. A tuple (a, b, b', a') representing the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "b"] \arrow[d, "a"'] & \cdot \arrow[d, ->>, "a'"] \\
\cdot \arrow[r, ->>, "b'"'] & \cdot
\end{tikzcd}$$

where $a, a', b, b' \in I^*$, is called **decreasing diagram** if $b \geq_{\mathrm{lmo}} b'/a$ and $a \geq_{\mathrm{lmo}} a'/b$.

Theorem: A labelled rewrite relation $(\to, I, >)$ is confluent if for all of its local peaks (a, b) have a decreasing diagram (a, b, b', a'). A rewrite relation $\to$ is confluent if there exists $(I, >)$ such that $(\to, I, >)$ is a confluent labelled rewrite relation.

Exercise 3 :
Prove Newman's lemma using decreasing diagrams techniques.

Solution:

Suppose $\to$ be a terminating and locally confluent rewrite relation on terms T. Since $\to$ is terminating, $\to^+$ is a well-founded order. We consider the labelled rewrite relation $(\to, T, \to^+)$ where each arrow $s \to t$ is labelled by s. As such, every local peak will have the form (s, s) for some $s \in T$. To prove the Lemma we just need to show that for every of such peaks there exists a decreasing diagram. Let (s, s) be the local peak corresponding to $s \to t'$ and $s \to t''$. Since $\to$ is locally confluent, it holds that there exists a term t such that $t' \to^* t$ and $t'' \to^* t$. Let b' and a' be the concatenation of labels corresponding to $t' \to^* t$ and $t'' \to^* t$ respectively. It holds that $s \to^+ \bar{t}$ for each term $\bar{t}$ in the path $t' \to^* t$ and in the path $t'' \to t$. As such, $\{\!|s|\!\}(\to^*)_{\mathrm{mul}}|b'|/\{\!|s|\!\} = \emptyset$ and $\{\!|s|\!\}(\to^*)_{\mathrm{mul}}|a'|/\{\!|s|\!\} = \emptyset$. We conclude that there exists a decreasing diagram from each local peak of $(\to, T, \to^+)$. By applying the Theorem above, $\to$ is therefore confluent.

Exercise 4 :

1. Let R be a left and right linear TRS. For every, $(l, r) \in \mathcal{R}$, define $t \xrightarrow{(l,r)} s$ iff $t \to_{\mathcal{R}} s$ using the rule (l, r). Assume given a well founded order $>$ on $\mathcal{R}$. Prove that if every critical peak of $(\to_{\mathcal{R}}, \mathcal{R}, >)$ has a decreasing diagram, then $\to_{\mathcal{R}}$ is confluent.
   This principle is called the **rule-labelling heuristic**.
2. Consider the following TRS:

$$\begin{aligned} \mathtt{nat} &\to \mathtt{0} : \mathtt{inc}(\mathtt{nat}) \\ \mathtt{inc}(x : y) &\to s(x) : \mathtt{inc}(y) \\ \mathtt{tl}(x : y) &\to y \\ \mathtt{inc}(\mathtt{tl}(\mathtt{nat})) &\to \mathtt{tl}(\mathtt{inc}(\mathtt{nat})) \end{aligned}$$

   Prove its confluence using the rule-labelling heuristic.
3. Consider the following (non-confluent) TRS $\mathcal{R}$:

$$\begin{aligned} \mathtt{a} &\to \mathtt{b} \\ \mathtt{b} &\to \mathtt{a} \\ \mathtt{a} &\to \mathtt{0} \\ \mathtt{b} &\to \mathtt{1} \end{aligned}$$

   Show that the rule-labelling heuristic cannot hold by proving that for any well-founded order $>$ on $\mathcal{R}$ there exists a critical peak with no decreasing diagrams.
4. Consider the following TRS $\mathcal{R}$:

$$\begin{aligned} \mathtt{a} &\to \mathtt{b} \\ \mathtt{p}(\mathtt{a}) &\to \mathtt{s}(\mathtt{p}(\mathtt{a})) \\ \mathtt{p}(\mathtt{b}) &\to \mathtt{r} \\ \mathtt{s}(x) &\to \mathtt{m}(x, x) \\ \mathtt{m}(x, y) &\to \mathtt{r} \end{aligned}$$

   Why the rule-labelling heuristic cannot be applied? Is it possible to find a set of labels I and a well-founded order $>$ on it such that every local peak of $(\to_{\mathcal{R}}, I, >)$ has a decreasing diagram?

Solution:

(1) If $\mathcal{R}$ is left and right linear, then for each local peak (a, b) that is not a critical peak will hold that one of (a, b, b, ϵ), (a, b, b, a) or (a, b, ϵ, a) is a decreasing diagram. Indeed, all three tuples satisfy $|b| \geq_{\mathrm{mul}} |b'|/|a|$ and $|a| \geq_{\mathrm{mul}} |a'|/|b|$, so it's only sufficient to show that the labelled rewrite relation $(\to_{\mathcal{R}}, \mathcal{R}, >)$ leads to one of those diagrams. This can be easily shown by considering the positions where the rewrite rules are applied. For example, consider $s \xrightarrow{(l,r)} t$ and $s \xrightarrow{(l',r')} t$ where (l, r) and (l', r') are applied in incomparable positions, i.e. there

exists two positions p, p' and two substitutions σ, σ' such that $p \not\leq p'$, $p' \not\leq p$, $s|_p = l\sigma$, $s|_{p'} = l'\sigma'$ and $t = s[r\sigma]_p$ and $t' = s[r'\sigma']_{p'}$. Then from $p \not\leq p'$ and $p' \not\leq p$ it holds that $t[r'\sigma']_{p'} = t'[r\sigma]_p = s[r\sigma]_p[r'\sigma']_{p'}$. Therefore $((l, r), (l', r'), (l', r'), (l, r))$ is a decreasing diagram for $((l, r), (l', r'))$. A similar analysis can be done for non-critical local peak resulting from two rewrite rules applied in two positions p, p' where $p < p'$ (in this case is important to use the hypothesis of left and right linearity).

Since any non-critical local peak (a, b) has a decreasing diagram, if also every critical peak of $(\rightarrow_{\mathcal{R}}, \mathcal{R}, >)$ has a decreasing diagram, then by Theorem it follows that $\rightarrow_{\mathcal{R}}$ is confluent.

(2) The TRS is left and right linear. We refer to its rules as follows

$$\begin{aligned}
1:&\quad \mathtt{nat} \rightarrow \mathtt{0} : \mathtt{inc}(\mathtt{nat}) \\
2:&\quad \mathtt{inc}(x : y) \rightarrow s(x) : \mathtt{inc}(y) \\
3:&\quad \mathtt{tl}(x : y) \rightarrow y \\
4:&\quad \mathtt{inc}(\mathtt{tl}(\mathtt{nat})) \rightarrow \mathtt{tl}(\mathtt{inc}(\mathtt{nat}))
\end{aligned}$$

The only critical pair arises from the term $\mathtt{tl}(\mathtt{inc}(\mathtt{nat}))$ and leads to the critical peak $(4, 1)$. The diagram $(4, 1, 123, 3)$ is decreasing by considering any ordering where 4 is greater than 1, 2 and 3. For example lets consider $4 > 3 > 2 > 1$. It holds that $\{\!\{3\}\!\} = |3| \geq_{\mathrm{mul}} |123|/|4| = \emptyset$ and $\{\!\{4\}\!\} = |4| \geq_{\mathrm{mul}} |3|/|1| = \{\!\{3\}\!\}$.

(3) We will refer to the rules of the TRS as: 1 ($\mathtt{a} \rightarrow \mathtt{b}$), 2 ($\mathtt{b} \rightarrow \mathtt{a}$), 3 ($\mathtt{a} \rightarrow \mathtt{0}$), 4 ($\mathtt{b} \rightarrow \mathtt{1}$). The TRS is therefore represented by the following diagram.

$$\begin{tikzcd}
\mathtt{0} & \mathtt{a} \arrow[l, "3"'] \arrow[r, bend left, "1"] & \mathtt{b} \arrow[l, bend left, "2"] \arrow[r, "4"] & \mathtt{1}
\end{tikzcd}$$

The TRS has two critical peaks $(1, 3)$ and $(2, 4)$. Let's consider the first one: after doing a 1 transition we can reach the irreducible state $\mathtt{0}$ only with paths of the form $2(12)^*3$. We conclude that the diagrams for the critical peak $(1, 3)$ can be characterized by $\{(1, 3, 2(12)^n 3, \epsilon) \mid n \in \mathbb{N}\}$. It must therefore hold that $1 > 2$, otherwise for all n it will not hold that $|3| \geq_{\mathrm{mul}} |2(12)^n 3|/|1|$. Similarly, by considering the second critical peak, we conclude that after doing a transition 2 we can reach the irreducible state $\mathtt{1}$ only with paths of the form $1(21)^*4$. Therefore, the diagrams of the critical peak $(2, 4)$ can be characterized by $\{(2, 4, 1(21)^n 4, \epsilon) \mid n \in \mathbb{N}\}$. As such, it must hold that $2 > 1$, otherwise for all $n \in \mathbb{N}$ it will not hold that $|4| \geq_{\mathrm{mul}} |1(21)^n 4|/|2|$. We conclude that it does not exists a well founded order $>$ on the TRS such that every of its critical peaks have a decreasing diagram for it.

(4) The rule-labelling heuristic requires the TRS to be right-linear, which is not the case here. In particular, the rule $\mathtt{s}(x) \rightarrow \mathtt{m}(x, x)$ can "self-duplicate". For example consider the term $\mathtt{s}(\mathtt{s}(\mathtt{r}))$. Still, it is easy to find a decreasing labelling noting that the duplicated variable has on the right-hand side less $\mathtt{s}$ symbols above it than on it left-hand side. Therefore, by labelling the steps first by the number of $\mathtt{s}$ symbols above the term and then by the rule, we can prove that any local peak has a decreasing diagram by considering the lexicographic order on these new labels.