Rewriting Techniques: TD 4

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A relation $\to$ is **locally confluent** if and only if for all r, s, t: $r \leftarrow s \to t \implies r \downarrow t$.

A **unification problem** P consist in a set of equations $s =^? t$ between terms. A solution of P is a substitution σ such that, for every equation $s =^? t$ in P we have $s\sigma = t\sigma$. We say that two terms s, t are unifiable if the unification problem $\{s =^? t\}$ has a solution. If two terms are unifiable then there exists a smallest solution, called **most general unifier** (mgu), w.r.t. the pointwise instantiation quasi-order.

Exercise 1 :
Prove that the most general unifier is unique up to renaming.

Solution:
W.l.o.g. we can assume only one equation $s =^? t$. Let σ_1, σ_2 be two most general unifiers for $s =^? t$. From the definition of mgu, it holds that a unifier $\overline{\sigma}$ is a mgu for s and t if and only if for all unifiers σ there exists σ' such that $s\sigma = (s\overline{\sigma})\sigma'$ and $t\sigma = (t\overline{\sigma})\sigma'$. Therefore, since σ_1 and σ_2 it must hold that there exists two unifiers r, r' such that

$$\begin{aligned} s\sigma_1 &= (s\sigma_2)r \\ t\sigma_1 &= (t\sigma_2)r \\ s\sigma_2 &= (s\sigma_1)r' \\ t\sigma_2 &= (t\sigma_1)r' \end{aligned}$$

in particular it must therefore hold that $s\sigma_1 = ((s\sigma_1)r)r'$ that holds if and only if r and r' are variable renaming. We conclude that σ_2 is a renaming of σ_1 (and vice-versa) and therefore the mgu is unique up to renaming.

Let $l \to r$, $l' \to r'$ be two rules whose variables have been renamed such that $(Var(l) \cup Var(r)) \cap (Var(l') \cup Var(r')) = \emptyset$. Let $p \in \mathrm{Pos}(l)$ be such that $l|_p$ is not a variable and let σ be an mgu of $l|_p =^? l'$. This determines a **critical pair** $(r\sigma, (l\sigma)[r'\sigma]_p)$. We are interested in **non-trivial** critical pairs, i.e critical pairs (s, t) where $s \neq t$.

Exercise 2 :
Compute the critical pairs of the following rewrite systems. Which one are locally confluent?

1. $\mathtt{s}(\mathtt{p}(\mathtt{s}(y))) \to y$, $\mathtt{s}(\mathtt{p}(x)) \to \mathtt{p}(\mathtt{s}(x))$
2. $0 + y \to y$, $x + 0 \to x$, $\mathtt{s}(w) + z \to \mathtt{s}(w + z)$, $v + \mathtt{s}(k) \to \mathtt{s}(v + k)$
3. $\mathtt{a}(x, x) \to \mathtt{0}$, $\mathtt{a}(y, \mathtt{p}(y)) \to \mathtt{1}$
4. $\mathtt{a}(\mathtt{a}(x, y), z) \to \mathtt{a}(x, \mathtt{a}(y, z))$, $\mathtt{a}(w, \mathtt{1}) \to w$

Solution:
(1) The critical pairs are determined from the substitutions $\sigma_1 = [x/\mathtt{s}(y)]$ and $\sigma_2 = [y/\mathtt{p}(x)]$ and $\sigma_3 = [y/\mathtt{p}(\mathtt{s}(z))]$, where σ_3 is obtained considering $\mathtt{s}(\mathtt{p}(\mathtt{s}(y)))$ with its renaming $\mathtt{s}(\mathtt{p}(\mathtt{s}(z)))$. From these three substitutions we get respectively the following three diagrams:

$$
\begin{tikzcd}[column sep=small]
 & \mathtt{s}(\mathtt{p}(\mathtt{s}(y))) \arrow[dl] \arrow[dr] & \\
y & & \mathtt{p}(\mathtt{s}(\mathtt{s}(y)))
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathtt{s}(\mathtt{p}(\mathtt{s}(\mathtt{p}(x)))) \arrow[d] \arrow[dr] \arrow[ddl] & \\
 & \mathtt{s}(\mathtt{p}(\mathtt{p}(\mathtt{s}(x)))) \arrow[d] & \mathtt{p}(\mathtt{s}(\mathtt{s}(\mathtt{p}(x)))) \arrow[dl] \\
\mathtt{p}(x) & \mathtt{p}(\mathtt{s}(\mathtt{p}(\mathtt{s}(x)))) \arrow[l] \arrow[d] & \\
 & \mathtt{p}(\mathtt{p}(\mathtt{s}(\mathtt{s}(x)))) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathtt{s}(\mathtt{p}(\mathtt{s}(\mathtt{p}(\mathtt{s}(z))))) \arrow[dl] \arrow[dr] & \\
\mathtt{s}(\mathtt{p}(z)) \arrow[rr] & & \mathtt{p}(\mathtt{s}(z))
\end{tikzcd}
$$

where the critical pairs are $(\mathtt{s}(\mathtt{p}(\mathtt{s}(y))), y)$, $(\mathtt{p}(x), \mathtt{s}(\mathtt{p}(\mathtt{p}(\mathtt{s}(x)))))$, $(\mathtt{p}(x), \mathtt{p}(\mathtt{s}(\mathtt{s}(\mathtt{p}(x)))))$ and $(\mathtt{s}(\mathtt{p}(z)), \mathtt{p}(\mathtt{s}(z)))$. From the diagrams it follows that the TRS is not locally confluent.

(2) The critical pairs are determined from the substitutions $\sigma_{1,2} = [x/\mathtt{0}, y/\mathtt{0}]$, $\sigma_{1,4} = [v/\mathtt{0}, y/\mathtt{s}(k)]$, $\sigma_{2,3} = [x/\mathtt{s}(w), z/\mathtt{0}]$, $\sigma_{3,4} = [v/\mathtt{s}(w), z/\mathtt{s}(k)]$., where $\sigma_{i,j}$ is defined from the rules i and j. From these substitutions we get the following diagrams:

$$
\begin{tikzcd}
\mathtt{0}+\mathtt{0} \arrow[d] \\
\mathtt{0}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathtt{0}+\mathtt{s}(k) \arrow[dl] \arrow[dr] & \\
\mathtt{s}(\mathtt{0}+k) \arrow[rr] & & \mathtt{s}(k)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathtt{s}(w)+\mathtt{0} \arrow[dl] \arrow[dr] & \\
\mathtt{s}(w+\mathtt{0}) \arrow[rr] & & \mathtt{s}(w)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathtt{s}(w)+\mathtt{s}(k) \arrow[dl] \arrow[dr] & \\
\mathtt{s}(w+\mathtt{s}(k)) \arrow[dr] & & \mathtt{s}(\mathtt{s}(w)+k) \arrow[dl] \\
 & \mathtt{s}(\mathtt{s}(w+k)) &
\end{tikzcd}
$$

where the critical pairs are $(\mathtt{0}, \mathtt{0})$, $(\mathtt{s}(\mathtt{0}+k), \mathtt{s}(k)$, $(\mathtt{s}(w+\mathtt{0}), \mathtt{s}(w))$, $(\mathtt{s}(w+\mathtt{s}(k)), \mathtt{s}(\mathtt{s}(w)+k))$. From the diagrams it follows that the TRS is locally confluent (actually stronger than that: the diamond property is satisfied).

(3) There are no non-trivial critical pairs since the unification problem $y =^? \mathtt{p}(y)$ does not admit any solution. Therefore, the TRS is locally confluent.

(4) The critical pairs are determined from the substitutions $\sigma_1 = [w/\mathtt{a}(x,y), z/\mathtt{1}]$, $\sigma_2 = [x/w, y/\mathtt{1}]$ and $\sigma_3 = [x/\mathtt{a}(x',y'), y/z']$, where σ_3 is obtained considering $\mathtt{a}(\mathtt{a}(x,y),z)$ with its renaming $\mathtt{a}(\mathtt{a}(x',y'),z')$. From these three substitutions we get respectively the following diagrams:

$$
\begin{tikzcd}[column sep=small]
 & \mathtt{a}(\mathtt{a}(x,y),\mathtt{1}) \arrow[dl] \arrow[dr] & \\
\mathtt{a}(x,\mathtt{a}(y,\mathtt{1})) \arrow[rr] & & \mathtt{a}(x,y)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathtt{a}(\mathtt{a}(w,\mathtt{1}),z) \arrow[dl] \arrow[dr] & \\
\mathtt{a}(w,z) & & \mathtt{a}(w,\mathtt{a}(\mathtt{1},z))
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \mathtt{a}(\mathtt{a}(\mathtt{a}(x',y'),z'),z) \arrow[dl] \arrow[dr] & \\
\mathtt{a}(\mathtt{a}(x',\mathtt{a}(y',z')),z) \arrow[d] & & \mathtt{a}(\mathtt{a}(x',y'),\mathtt{a}(z',z)) \arrow[d] \\
\mathtt{a}(x',\mathtt{a}(\mathtt{a}(y',z'),z)) \arrow[rr] & & \mathtt{a}(x',\mathtt{a}(y',\mathtt{a}(z',z)))
\end{tikzcd}
$$

where the critical pairs are $(\mathtt{a}(x,\mathtt{a}(y,\mathtt{1})), \mathtt{a}(x,y))$ and $(\mathtt{a}(w,z), \mathtt{a}(w,\mathtt{a}(\mathtt{1},z)))$. From the diagrams it follows that the TRS is not locally confluent.

A **completion procedure** is a program that accepts as input a finite set of identities E_0 and a reduction order $>$, and generate a (finite or infinite) sequence (called run) $(E_0, R_0), (E_1, R_1), (E_2, R_2), \ldots$

where $R_0 = \emptyset$, by applying the rules:

$$\frac{E, R \quad s \leftarrow_R u \rightarrow_R t}{E \cup \{s \approx t\}, R}\ \text{Deduce} \qquad\qquad \frac{E \cup \{s \dot{\approx} t\}, R \quad s > t}{E, R \cup \{s \rightarrow t\}}\ \text{Orient}$$

$$\frac{E \cup \{s \approx s\}, R}{E, R}\ \text{Delete} \qquad\qquad \frac{E \cup \{s \dot{\approx} t\}, R \quad s \rightarrow_R u}{E, R \cup \{u \rightarrow t\}}\ \text{Simplify-Id}$$

$$\frac{E, R \cup \{s \rightarrow t\} \quad t \rightarrow_R u}{E, R \cup \{s \rightarrow u\}}\text{R-Simplify} \qquad\qquad \frac{E, R \cup \{s \rightarrow t\} \quad s \stackrel{\sqsupset}{\rightarrow}_R u}{E \cup \{u \approx t\}, R}\text{L-Simplify}$$

where $s \dot{\approx} t$ if and only if $s \approx t$ or $t \approx s$, whereas $s \stackrel{\sqsupset}{\rightarrow}_R u$ is used to express that s is reduced to u by a rule $l \rightarrow r$ of R such that each subterm of l is not an instance of s. A special case of *Deduce* is to apply it only if (s, t) is a critical pair. Most completion procedures use the rule *Deduce* only in this way. The goal of these procedures is to teransform an initial pair $(E_0, \emptyset)$ into a pair $(\emptyset, \mathcal{R})$ such that $\mathcal{R}$ is a convergent TRS equivalent to E_0.

Algorithm 1 Huet's completion procedure

Require: A finite set E of identities and a reduction order $>$

Ensure: A finite convergent (terminating and confluent) rewrite system R equivalent to E if the procedure terminates successfully, FAIL if the procedure terminates unsuccessfully

1: $R_0 := \varnothing$; $E_0 := E$; $i := 0$; all identities of E are unmarked
2: **while** $E_i \neq \varnothing$ or there is an unmarked rule in R_i **do**
3:   **while** $E_i \neq \varnothing$ **do**
4:     Choose an identity $(s, t) \in E$ and reduce s and t to some R_i-normal forms $\tilde{s}$ and $\tilde{t}$
5:     **if** $\tilde{s} = \tilde{t}$ **then**
6:       $R_{i+1} := R_i$; $E_{i+1} := E_i \setminus \{(s, t)\}$; $i := i + 1$
7:     **else if** $\tilde{s} \not> \tilde{t} \wedge \tilde{t} \not> \tilde{s}$ **then**
8:       terminates with output FAIL
9:     **else**
10:       let l and r such that $\{l, r\} = \{\tilde{s}, \tilde{t}\}$ and $l > r$
11:       $R_{i+1} := \{(g, \tilde{d}) \mid (g, d) \in R_i \wedge g$ cannot be reduced with $l \rightarrow r \wedge \tilde{d}$ is a $R_i \cup \{(l, r)\}$-normal form of $d\} \cup \{(l, r)\}$
12:       (l, r) is not marked and $(g, \tilde{d})$ is marked in R_{i+1} iff (g, d) is marked in R_i
13:       $E_{i+1} := (E_i \setminus \{(s, t)\}) \cup \{(g', d) \mid (g, d) \in R_i \wedge g$ can be reduced to g' with $l \rightarrow r\}$
14:       $i := i + 1$
15:     **end if**
16:   **end while**
17:   **if** there is an unmarked rule in R_i **then**
18:     let (l, r) be such a rule
19:     $R_{i+1} := R_i$
20:     $E_{i+1} := \{(s, t) \mid (s, t)$ is a critical pair of (l, r) with itself or with a marked rule in $R_i\}$
21:     Mark (l, r) ; $i := i + 1$
22:   **end if**
23: **end while**
24: **return** R_i

Exercise 3 :
Show that Huet's completion procedure is indeed a completion procedure.

Solution:

The computation of critical pairs in the outer while-loop can be realized using *Deduce*. The step at line 4 can be achieved using *Simplify-Id*, whereas lines 5 and 6 correspond to *Delete*. The steps of lines 10-13 can be achieved using *Orient* (line 10), *R-Simplify* (line 11) and *L-Simplify* (line 13). This is trivial for the orientation step. In order to show that *R-Simplify* can be used to generate rules $g \rightarrow \tilde{d} \in R_{i+1}$, one must prove that the simultaneous reductions $d \rightarrow^*_{R_i} \tilde{d}$, which are all done with the original system R_i, can be realized by a sequence of *R-Simplify* steps, in which the already partially modified TRS must be used in each simplification step.

Lastly, note that the $\sqsupset$ condition of *L-Simplify* is satisfied when reducing the left-hand side of a rule $g \to d \in R_i$ with $l \to r$ since l is in R_i-normal form and thus any subterm of l cannot be an instance of g.

Some notions from the second lecture:

A strict order $>$ on $T(\Sigma, V)$ is called a **rewrite order** if and only iff

1. is *compatible*: for all $s_1, s_2 \in T(\Sigma, V)$, all $f \in \Sigma$, if $s_1 > s_2$ then

$$f(t_1, \ldots, t_{i-1}, s_1, t_{i+1}, \ldots, t_n) > f(t_1, \ldots, t_{i-1}, s_2, t_{i+1}, \ldots, t_n)$$

   where n is the arity of f;
2. is *closed under substitution*: for all $s_1, s_2 \in T(\Sigma, V)$ and all substitutions $\sigma : V \to T(\Sigma, V)$, if $s_1 > s_2$ then $\sigma(s_1) > \sigma(s_2)$.

A **reduction order** is a well-founded rewrite order. LPO and KBO (defined below) are two examples of reduction orders.

The **lexicographic path order** (LPO) $>_{lpo}$ on $T(\Sigma, V)$ induced by a strict order $>$ on Σ is defined as follows. $s >_{lpo} t$ if and only if one of the following holds:

1. t is a variable appearing in s and $s \neq t$, or

let $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$,

2. there exists $i \in [1, m]$ such that $s_i \geq_{lpo} t$, or
3. $f > g$ and $s >_{lpo} t_j$ for all $j \in [1, n]$, or
4. $f = g$, for all $j \in [1, n]$ it holds $s >_{lpo} t_j$ and $(s_1, \ldots, s_m)(>_{lpo})_{\text{lex}}(t_1, \ldots, t_m)$.

where $(>_{lpo})_{\text{lex}}$ is the lexicographic w.r.t. $>_{lpo}$.

Let $>$ be a strict order on Σ and $w : \Sigma \cup V \to \mathbb{R}_0^+$ be a weight function $w : \Sigma \cup V \to \mathbb{R}_0^+$. The **Knuth-Bendix order** (KBO) $>_{\text{kbo}}$ on $T(\Sigma, V)$ induced by $>$ and w is defined as follows: for $s, t \in T(\Sigma, V)$ we have $s >_{\text{kbo}} t$ if and only if $|s|_x \geq |t|_x$ for all $x \in V$ and $w(s) \geq w(t)$. Moreover, if $w(s) = w(t)$ then one of the following properties must hold:

1. There are a unary function f, $x \in V$ and $n \in \mathbb{N}^{\geq 1}$ s.t. $s = f^n(x)$ and $t = x$, or
2. there exist function symbols f, g s.t. $f > g$ and $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$, or
3. there exist a function symbol f such that $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$ and

$$(s_1, \ldots, s_m)(>_{\text{kbo}})_{\text{lex}}(t_1, \ldots, t_m).$$

A weight function $w : \Sigma \cup V \to \mathbb{R}_0^+$ is called **admissible** if and only if it satisfy the following properties w.r.t. a strict order $>$:

1. There exists $w_0 \in \mathbb{R}_0^+ \backslash \{0\}$ s.t. $w(x) = w_0$ for all $x \in V$ and $w(c) \geq w_0$ for all constants $c \in \Sigma$.
2. If $f \in \Sigma$ is a unary function symbol of weight $w(f) = 0$ then f is the greatest element in Σ, i.e. $f \geq g$ for all $g \in \Sigma$.

Exercise 4 :
Apply the Huet's completion procedure on the following set of identities, with the suitable reduction order:

1. $\{(x * (y + z), (x * y) + (x * z)), ((u + v) * w, (u * w) + (v * w))\}$ and $>$ the LPO with $* > +$.
2. $\{(x + 0, x), (z + \mathsf{s}(y), \mathsf{s}(z + y))\}$ and $>$ the KBO with $\mathsf{s} > +$ and weight 1 for all variables and symbols. Consider then the KBO with $+ > \mathsf{s}$ and weight 1 for all variables and symbols.
3. $\{(f(g(f(x))), x)\}$ and the LPO with $f > g$.

Solution:

In the following solution, we only consider non-trivial critical pairs.

(1) From the LPO it holds that $x*(y+z) >_{lpo} (x*y)+(x*z)$ and $(u+v)*w >_{lpo} (u*w)+(v*w)$. $E_0 = \{(x*(y+z),(x*y)+(x*z)),((u+v)*w,(u*w)+(v*w))\}$ and $R_0 = \emptyset$. Select the first identity, that is already in normal form w.r.t. R_0. The procedure will compute $E_1 = \{((u+v)*w,(u*w)+(v*w))\}$ and $R_1 = \{(x*(y+z),(x*y)+(x*z))\}$. Select now the only identity in E_1, which is already in normal form w.r.t. R_1. $x*(y+z)$ cannot be reduced using $(u+v)*w$ (line 13) and therefore $R_2 = \{(x*(y+z),(x*y)+(x*z)),((u+v)*w,(u*w)+(v*w))\}$ and $E_2 = \emptyset$. Both elements in R_2 are not marked and therefore the procedure will continue by selecting one of these two identities (line 21). Since each identity does not have critical pairs with itself, and R_2 does not have any marked rules, it will holds (for example if the first identity is selected) $R_3 = R_2$, $E_3 = E_2$ and the first identity is marked. Now the procedure will select the second one (line 21) and compute the critical pair between that identity and the first rule. The only critical pair between $x*(y+z)$ and $(u+v)*w$ is determined by the substitution $\sigma = [x/u+v, w/y+z]$ which leads to the following diagram

$$\begin{tikzcd}
 & (u+v)*(y+z) \arrow[dl] \arrow[dr] & \\
((u+v)*y)+((u+v)*z) \arrow[d] & & (u*(y+z))+(v*(y+z)) \arrow[d] \\
((u*y)+(v*y))+((u*z)+(v*z)) & & ((u*y)+(u*z))+((v*y)+(v*z))
\end{tikzcd}$$

where the two terms $((u+v)*y)+((u+v)*z)$ and $(u*(y+z))+(v*(y+z))$ from a critical pair whereas the two terms $((u*y)+(v*y))+((u*z)+(v*z))$ and $((u*y)+(u*z))+((v*y)+(v*z))$ are their normal form. So $R_4 = \{(x*(y+z),(x*y)+(x*z)),((u+v)*w,(u*w)+(v*w))\}$ whereas $E_4 = \{(((u+v)*y)+((u+v)*z),(u*(y+z))+(v*(y+z)))\}$, with both identities of R_4 marked. The procedure will then evaluate the loop starting at line 3 by selecting $(((u+v)*y)+((u+v)*z),(u*(y+z))+(v*(y+z)))$ and reducing it to its normal form $(\tilde{s},\tilde{t})$. It holds that $\tilde{s} \neq \tilde{t}$, $\tilde{s} \not>_{lpo} \tilde{t}$ and $\tilde{t} \not>_{lpo} \tilde{s}$ and the condition at line 9 will be verified and the procedure will terminate with output FAIL.

(2) From the KBO (herein $>_{\text{kbo}}$) it holds that $\mathtt{s}(z+y) >_{\text{kbo}} z+\mathtt{s}(y)$ and $x+\mathtt{0} >_{\text{kbo}} x$. Skipping some easy steps, it will holds

$$R_2 = \{(\mathtt{s}(z+y), z+\mathtt{s}(y)),(x+\mathtt{0},x)\} \text{ and } E_2 = \emptyset$$

The only critical pair of R_2 is determined by the substitution $\sigma = [y/\mathtt{0}, z/x]$ which leads to the following diagram

$$\begin{tikzcd}
 & \mathtt{s}(x+\mathtt{0}) \arrow[dl] \arrow[dr] & \\
x+\mathtt{s}(\mathtt{0}) & & \mathtt{s}(x)
\end{tikzcd}$$

where $x+\mathtt{s}(\mathtt{0})$ and $\mathtt{s}(x)$ form a critical pair and it holds $x+\mathtt{s}(\mathtt{0}) >_{\text{kbo}} \mathtt{s}(x)$. Therefore, the procedure will define $R_3 = R2$ and $E_3 = \{(x+\mathtt{s}(\mathtt{0}),\mathtt{s}(x))\}$, where the identities of R_3 are marked. The procedure will then compute $R_4 = R_3 \cup E_3$ and $E_4 = \emptyset$. The identity $(x+\mathtt{s}(\mathtt{0}),\mathtt{s}(x))$ in R_4 is the only one not marked and therefore we need to compute its critical pairs w.r.t. itself and the marked identities (i.e. all the identities in R_3. There is one new critical pair between $\mathtt{s}(z+y)$ and $x+\mathtt{s}(\mathtt{0})$, correspondent to the substitution $\sigma' = [y/\mathtt{s}(\mathtt{0}), z/x]$. The critical pair is $(x+\mathtt{s}(\mathtt{s}(\mathtt{0})),\mathtt{s}(\mathtt{s}(x))$ and it holds that $x+\mathtt{s}(\mathtt{s}(\mathtt{0})) >_{\text{kbo}} \mathtt{s}(\mathtt{s}(x))$. The pair $(x+\mathtt{s}(\mathtt{s}(\mathtt{0})),\mathtt{s}(\mathtt{s}(x)))$ will therefore be the only element of E_5, whereas $R_5 = R_4$. The procedure will then evaluate again the while loop of line 3, resulting in $R_6 = R_5 \cup E_5$.

In general, we can show that the following invariant will always hold for $n \geq 2$ at line 20.

- $R_{2n} = \{(\mathtt{s}(z,y), z+\mathtt{s}(y)),(x+\mathtt{0},x)\} \cup \{(x+\mathtt{s}^k(\mathtt{0}),\mathtt{s}^k(x)) \mid k \leq n-1\}$;

- $E_{2n} = \emptyset$;
- all identities in R_{2n} but $(x + \mathtt{s}^{n-1}(\mathtt{0}), \mathtt{s}^{n-1}(x))$ are marked;
- $(x+\mathtt{s}^{n}(\mathtt{0}), \mathtt{s}^{n}(x))$ is a critical pair between $(x+\mathtt{s}^{n-1}(\mathtt{0}), \mathtt{s}^{n-1}(x))$ and $(\mathtt{s}(z, y), z+\mathtt{s}(y))$.

Therefore the procedure will not terminate. Instead, by considering the KBO with $+ >_{\text{kbo}} \mathtt{s}$ and weight 1 for all variables and symbols it will hold that $\mathtt{s}(z+y) < z+\mathtt{s}(y)$ and $x+\mathtt{0} >_{\text{kbo}} x$ and $R_2 = \{(z+\mathtt{s}(y), \mathtt{s}(z+y)), (x+\mathtt{0}, x)\}$. R_2 does not have any critical pairs and therefore is the convergent rewrite system returned by the procedure.

(3) Trivially $R_1 = \{(f(g(f(x))), x)\}$, $E_1 = \emptyset$ and $f(g(f(x))) >_{lpo} x$ w.r.t. the LPO $>_{lpo}$. The only critical pair of R_1 is determined by the substitution $\sigma = [x/g(f(z))]$ obtained considering $f(g(f(x)))$ with its renaming $f(g(f(z)))$. σ leads to the following diagram

$$\begin{tikzcd} & f(g(f(g(f(z))))) \arrow[dl] \arrow[dr] & \\ g(f(z)) & & f(g(z)) \end{tikzcd}$$

Where $(g(f(z)), f(g(z)))$ form a critical pair. The two term of these pair are already in normal form and it holds $f(g(z)) >_{lpo} g(f(z))$. Therefore $R_2 = \{(f(g(f(x))), x)\}$ whereas $E_2 = \{(f(g(z)), g(f(z)))\}$. The identity $(f(g(z)), g(f(z)))$ will then be selected (line 4). The procedure will enter the else branch in line 12 where $(f(g(f(x))), x)$ will be reduced by $(f(g(z)), g(f(z)))$ to $(g(f(f(x))), x)$. It will therefore hold $E_3 = \{(g(f(f(x))), x)\}$ and $R_3 = \{(f(g(z)), g(f(z)))\}$. The procedure will then select the only element in E_3. It holds $g(f(f(x))) >_{lpo} x$ and this identity cannot be used to reduce $(f(g(z)), g(f(z)))$. Therefore $R_4 = R_3 \cup E_3$, $E_4 = \emptyset$ and both rules in R_4 are not marked. The procedure will then mark one of the two rules in $R_4 = \{(g(f(f(x))), x), (f(g(z)), g(f(z)))\}$ and then check the critical pairs between the unmarked rule and the marked one. There are two critical pairs determined from the substitutions $\sigma_1' = [z/f(f(x))]$ and $\sigma_2' = [x/g(z)]$. From these substitutions we get the following diagrams:

$$\begin{tikzcd} & f(g(f(f(x)))) \arrow[dl] \arrow[dr] & \\ g(f(f(f(x)))) \arrow[rr] & & f(x) \end{tikzcd} \qquad \begin{tikzcd} g(f(f(g(z)))) \arrow[d] \arrow[ddd, bend right] & \\ g(f(g(f(z)))) \arrow[d] & \\ g(g(f(f(z)))) \arrow[d] & \\ g(z) & \end{tikzcd}$$

The two critical pairs $(g(f(f(f(x)))), f(x))$ and $(g(f(g(f(z)))), g(z))$ will then be added to E_5, whereas $R_5 = R_4 = \{(g(f(f(x))), x), (f(g(z)), g(f(z)))\}$. The procedure will then select the identities in E_5 but, since the two diagrams shown are confluent, it holds that the normal form $(\tilde{s}, \tilde{t})$ of each identity is such that $\tilde{s} = \tilde{t}$. Therefore (line 5) the procedure will remove the elements from E without any update on R. Lastly, since all elements of R_5 were already marked, the procedure will successfully return $\{(g(f(f(x))), x), (f(g(z)), g(f(z)))\}$.

Exercise 5 :

1. Prove that the set of identities

$$\begin{aligned} &(@(\mathtt{nil}, x),\ x),\\ &(@(\mathtt{cons}(x, y), z),\ \mathtt{cons}(x, @(y, z))),\\ &(\mathtt{rev}(\mathtt{nil}),\ \mathtt{nil}),\\ &(\mathtt{rev}(\mathtt{cons}(x, y)),\ @(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil}))) \end{aligned}$$

can be oriented to give a convergent TRS. Let R this TRS.

2. Why the associativity A of @, $@(@(x,y),z) = @(x,@(y,z))$ is not a consequence of R?
3. Prove that you can complete (A, R). You can use Huet's completion procedure.
4. Show that Huet's completion fails to complete $(\{\texttt{rev}(x) = @(x,x)\}, R)$.

Solution:

(1) We use LPO w.r.t. the order $\texttt{rev} > @ > \texttt{cons} > \texttt{nil}$ to orient the rules as follows:

$$
\begin{aligned}
@(\texttt{nil}, x) &>_{lpo} x \\
@(\texttt{cons}(x,y),z) &>_{lpo} \texttt{cons}(x, @(y,z)) \\
\texttt{rev}(\texttt{nil}) &>_{lpo} \texttt{nil} \\
\texttt{rev}(\texttt{cons}(x,y)) &>_{lpo} @(\texttt{rev}(y), \texttt{cons}(x,\texttt{nil}))
\end{aligned}
$$

$R = \{@(\texttt{nil},x) \to x,\ @(\texttt{cons}(x,y),z) \to \texttt{cons}(x,@(y,z)),\ \texttt{rev}(\texttt{nil}) \to \texttt{nil},\ \texttt{rev}(\texttt{cons}(x,y)) \to @(\texttt{rev}(y), \texttt{cons}(x,\texttt{nil}))\}$ is a TRS with no critical pairs. Therefore, by critical pairs Lemma, R is locally confluent. Moreover, since we can use LPO to prove its termination, by Newman's Lemma (which implies confluency of R), R is convergent.

Notice that if we change the orientation of the last rule to

$$@(\texttt{rev}(y), \texttt{cons}(x,\texttt{nil})) >_{lpo} \texttt{rev}(\texttt{cons}(x,y))$$

we obtain a TRS with a critical pair derived from the following diagram:

$$
\begin{tikzcd}
 & @(\texttt{rev}(\texttt{nil}), \texttt{cons}(x,\texttt{nil})) \arrow[dl] \arrow[dr] & \\
@(\texttt{nil}, \texttt{cons}(x,\texttt{nil})) \arrow[d] & & \texttt{rev}(\texttt{cons}(x,\texttt{nil})) \\
\texttt{cons}(x,\texttt{nil}) & &
\end{tikzcd}
$$

Which is not a convergent critical pair. Therefore this orientation is not enough to get a convergent TRS and we would need to apply a completion procedure.

(2) $@(@(x,y),z)$ and $@(x,@(y,z))$ are already in normal form w.r.t. R and therefore associativity is not a consequence of R.

(3) Consider the LPO of (1), E_0 equal to

$$
\begin{aligned}
&1 : (@(\texttt{nil},x),\ x), \\
&2 : (@(\texttt{cons}(x,y),z),\ \texttt{cons}(x,@(y,z))), \\
&3 : (\texttt{rev}(\texttt{nil}),\ \texttt{nil}), \\
&4 : (\texttt{rev}(\texttt{cons}(x,y)),\ @(\texttt{rev}(y), \texttt{cons}(x,\texttt{nil}))), \\
&5 : (@(@(x',y'),z'),\ @(x',@(y',z')))
\end{aligned}
$$

and $R_0 = \emptyset$. Notice that, as written here, E_0 is already oriented w.r.t. LPO, i.e. $E_0 \subseteq >_{lpo}$. Moreover, the first argument of each pair in E_0 cannot be reduced w.r.t. the TRS induced by E_0 (i.e. $l \to r$ is in the TRS iff $(l,r) \in E_0$). Also, the second argument of each pair in E_0 is already in normal form w.r.t. the TRS induced by E_0. As such, the while of line 3 of the Huet's procedure will simply move each element of E_0 to R_5. As such, consider $E_4 = \{(@(@(x',y'),z'),\ @(x',@(y',z')))\} = A$ and $R_4 = E_0 \setminus A = E$, which is what we want to compute in the exercise. The procedure will continue by moving also the fifth rule to R_5. So, $R_5 = E_0$ and $E_5 = \emptyset$. The procedure will continue by evaluating the conditional at line 20. As already shown in (1), without the last rule E does not have any critical pairs. As such, we only consider the critical pairs between the associativity rule and the other rules. We derive the following diagrams based on the substitutions $\sigma_{5,1} = [x'/\texttt{nil}]$, $\sigma_{5,2} = [x'/\texttt{cons}(x,y)]$ and $\sigma_{5,5} = [x'/@(x,y)]$, where $\sigma_{i,j}$ is the mgu that can be used to compute a critical pair between the rules i and j:

$$
\begin{tikzcd}[column sep=tiny]
 & @(@(\mathtt{nil}, y), z) \arrow[dl] \arrow[dr] & \\
@(\mathtt{nil}, @(y, z)) \arrow[rr] & & @(y, z)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & @(@(\mathtt{cons}(x, y), z), w) \arrow[dl] \arrow[dr] & \\
@(\mathtt{cons}(x, y), @(z, w)) \arrow[dr] & & @(\mathtt{cons}(x, @(y, z)), w) \arrow[dl] \\
 & \mathtt{cons}(x, @(y, @(z, w))) &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & @(@(@(x, y), z), w) \arrow[dl] \arrow[dr] & \\
@(@(x, y), @(z, w)) \arrow[ddrr] & & @(@(x, @(y, z)), w) \arrow[d] \\
 & & @(x, @(@(y, z), w)) \arrow[d] \\
 & & @(x, @(y, @(z, w)))
\end{tikzcd}
$$

The critical pairs in these diagrams are

- $(@(\mathtt{nil}, @(y, z)), @(y, z))$ where $@(\mathtt{nil}, @(y, z)) >_{lpo} @(y, z)$,
- $(@(\mathtt{cons}(x, y), @(z, w)), @(\mathtt{cons}(x, @(y, z)), w))$ where

$$@(\mathtt{cons}(x, y), @(z, w)) >_{lpo} @(\mathtt{cons}(x, @(y, z)), w)$$

- $(@(@(x, y)@(z, w)), @(@(x, @(y, z)), w)$ where $@(@(x, @(y, z)), w) >_{lpo} @(@(x, y)@(z, w))$.

Notice how all three diagrams are confluent. For this reason, lets put aside the Huet's procedure and consider the TRS:

$$
\begin{aligned}
@(\mathtt{nil}, x) &\to x \\
@(\mathtt{cons}(x, y), z) &\to \mathtt{cons}(x, @(y, z)) \\
\mathtt{rev}(\mathtt{nil}) &\to \mathtt{nil} \\
\mathtt{rev}(\mathtt{cons}(x, y)) &\to @(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil})) \\
(@(@(x', y'), z') &\to @(x', @(y', z')))
\end{aligned}
$$

its easy to show that this TRS is terminating w.r.t. the LPO induced by the order the order $\mathtt{rev} > @ > \mathtt{cons} > \mathtt{nil}$. Moreover, from the diagrams above, all critical pairs are convergent. As such, the TRS is locally confluent and by Newman's Lemma, is also convergent. Moreover, in this TRS it trivially holds that $(@(@(x', y'), z')$ and $@(x', @(y', z')))$ have the same normal form and the TRS is a completion for (A, R).

(4) For simplicity, let $R_0 = R$ and $E_0 = \{(\mathtt{rev}(x), @(x, x))\}$ and suppose we start the evaluation of Huet's procedure from line 3. It holds that $\mathtt{rev}(x) >_{lpo} @(x, x)$ and the two terms are already in R_0-normal form. The rules $(\mathtt{rev}(\mathtt{nil}), \mathtt{nil})$ and $(\mathtt{rev}(\mathtt{cons}(x, y)), @(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil})))$ can be reduced via $\mathtt{rev}(x)$. As such, it will hold that

$$R_1 = \{(@(\mathtt{nil}, x), x), (@(\mathtt{cons}(x, y), z), \mathtt{cons}(x, @(y, z))), (\mathtt{rev}(x), @(x, x))\}$$

and

$$E_1 = \{(@(\mathtt{nil}, \mathtt{nil}), \mathtt{nil}), (@(\mathtt{cons}(x, y), \mathtt{cons}(x, y)), @(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil})))\}$$

The first identity of R_1 will be simply removed since the R_1-normal form of $@(\mathtt{nil}, \mathtt{nil})$ is exactly $\mathtt{nil}$. Instead, the normal form of the second identity in E_1 is

$$(\mathtt{cons}(x, @(y, \mathtt{cons}(x, y))), @(@(y, y), \mathtt{cons}(x, \mathtt{nil})))$$

and is such that $@(@(y, y), \mathtt{cons}(x, \mathtt{nil})) >_{lpo} \mathtt{cons}(x, @(y, \mathtt{cons}(x, y)))$. Therefore, E and R will be updated so that $E_3 = \emptyset$ and

$$R_3 = \{(@(\texttt{nil}, x), x), (@(\texttt{cons}(x, y), z), \texttt{cons}(x, @(y, z))), (\texttt{rev}(x), @(x, x)),$$
$$(@(@(y, y), \texttt{cons}(x, \texttt{nil})), \texttt{cons}(x, @(y, \texttt{cons}(x, y))))\}$$

The procedure will then search for critical pairs of R_3 and eventually find the critical pairs between $@(\texttt{cons}(x, y), z)$ and $@(@(y', y'), \texttt{cons}(x', \texttt{nil}))$, in particular w.r.t. the substitution $\sigma = [y'/\texttt{cons}(x, y), z/\texttt{cons}(x, y)]$ from which we derive the following diagram

$$\begin{tikzcd}
 & @(@(\texttt{cons}(x,y), \texttt{cons}(x,y)), \texttt{cons}(x', \texttt{nil}))) \arrow[dl] \arrow[dr] & \\
@(\texttt{cons}(x, @(y, \texttt{cons}(x,y))), \texttt{cons}(x', \texttt{nil})) \arrow[d] & & \texttt{cons}(x', @(\texttt{cons}(x,y), \texttt{cons}(x', \texttt{cons}(x,y)))) \\
\texttt{cons}(x, @(@(y, \texttt{cons}(x,y)), \texttt{cons}(x', \texttt{nil}))) & &
\end{tikzcd}$$

the critical pair

$$(@(\texttt{cons}(x, @(y, \texttt{cons}(x, y))), \texttt{cons}(x', \texttt{nil})), \texttt{cons}(x', @(\texttt{cons}(x, y), \texttt{cons}(x', \texttt{cons}(x, y)))))$$

will be added to E and eventually selected in line 4. However, from the diagram we can see that the normal form of this critical pair is $(\tilde{s}, \tilde{t}) =$

$$(\texttt{cons}(x, @(@(y, \texttt{cons}(x, y)), \texttt{cons}(x', \texttt{nil}))), \texttt{cons}(x', @(\texttt{cons}(x, y), \texttt{cons}(x', \texttt{cons}(x, y)))))$$

and is such that $\tilde{s} \neq \tilde{t}$, $\tilde{s} \not>_{lpo} \tilde{t}$ and $\tilde{t} \not>_{lpo} \tilde{s}$ and therefore the completion procedure will return FAIL. The completion will also fail if we considered an order where $@(x, x) > \texttt{rev}(x)$, as implied by (1).

We recall that the **Post Correspondence Problem** (PCP) is an undecidable decision problem defined as follows: the input of the problem consists of a finite set of pairs of non-empty words $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_n, \beta_n)\}$ over some alphabet A, where $|A| \geq 2$. The problem has a solution whenever there is a sequence of indices $(i_k)_{1 \leq k \leq K}$ with $K \geq 1$ and $i_k \in [1, n]$ such that $\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_{k-1}}\alpha_{i_k} = \beta_{i_1}\alpha_{i_2}\dots\beta_{i_{k-1}}\beta_{i_k}$.

Exercise 6 :
Let $P = (\alpha_i, \beta_i)_{1 \leq i \leq n}$ be an instance of PCP, with $\alpha_i, \beta_i \in \{\texttt{0}, \texttt{1}\}^+$ for $i \in [1, n]$. Define

$$R(P) = \{A \to f(\alpha_i(\epsilon), \beta_i(\epsilon)), f(x, y) \to f(\alpha_i(x), \beta_i(y)), f(x, x) \to B, f(x, y) \to A\}$$

on $\mathcal{F} = \{f(2), A(0), B(0), \texttt{0}(1), \texttt{1}(1), \epsilon(0)\}$.

1. Prove that P has a solution iff $A \to^* B$.
2. Deduce that confluence is undecidable.

Solution:
(1) If P has a solution $i_1, \dots, i_k$ and therefore $\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k} = \beta_{i_1}\beta_{i_2}\dots\beta_{i_k}$. Then it holds that:

$$A \to f(\alpha_{i_k}(\epsilon), \beta_{i_k}(\epsilon)) \to f(\alpha_{i_{k-1}}\alpha_{i_k}(\epsilon), \beta_{i_{k-1}}\beta_{i_k}(\epsilon)) \to \dots$$
$$\to f(\alpha_{i_1}\dots\alpha_{i_{k-1}}\alpha_{i_k}(\epsilon), \beta_{i_1}\dots\beta_{i_{k-1}}\beta_{i_k}(\epsilon)) \to B$$

The converse is also easy, by looking at the rules applications that leads to B.

(2) Sufficient to notice that, thanks to the rule $f(x, y) \to A$, each term which is different from B will reach A. In particular, terms of the form $f(x, x)$ are such that $f(x, x) \to A$ and $f(x, x) \to B$.If $A \to^* B$ holds, then the TRS is confluent. Otherwise, since there is no path from A to B, which are both reduced from $f(x, x)$, the TRS won't be confluent. From the previous point, it therefore holds that PCP has a solution if and only if the TRS herein defined is confluent. It follows that confluence is undecidable.