Rewriting Techniques: TD 3

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A **unification problem** P consist in a set of equations $s =^? t$ between terms. A solution of P is a substitution σ such that, for every equation $s =^? t$ in P we have $s\sigma = t\sigma$. We say that two terms s, t are unifiable if the unification problem $\{s =^? t\}$ has a solution. If two terms are unifiable then there exists a smallest solution, called **most general unifier** (mgu), w.r.t. the pointwise instantiation quasi-order.

Exercise 1 :
Find (if it exists) the *mgu* of the following unification problems:

- $\mathtt{s}(\mathtt{p}(x, y)) =^? \mathtt{s}(\mathtt{p}(\mathtt{p}(\mathtt{s}(\mathtt{a}), y), \mathtt{s}(\mathtt{b})))$;
- $\mathtt{b}(x, \mathtt{a}, \mathtt{p}(\mathtt{s}(\mathtt{a}))) =^? \mathtt{b}(\mathtt{s}(y), y, \mathtt{p}(x))$;
- $\mathtt{p}(\mathtt{s}(y), \mathtt{s}(x)) =^? \mathtt{p}(x, \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a})))$;
- $\{\mathtt{s}(\mathtt{p}(x, z)) =^? \mathtt{s}(\mathtt{p}(\mathtt{p}(\mathtt{s}(\mathtt{a}), y), \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a})))),\ \mathtt{p}(y, z) =^? \mathtt{p}(x, \mathtt{s}(\mathtt{p}(\mathtt{a}, y)))\}$;

Solution:
(1)

$$\{\mathtt{s}(\mathtt{p}(x, y)) =^? \mathtt{s}(\mathtt{p}(\mathtt{p}(\mathtt{s}(\mathtt{a}), y), \mathtt{s}(\mathtt{b})))\} \to \{\mathtt{p}(x, y) =^? \mathtt{p}(\mathtt{p}(\mathtt{s}(\mathtt{a}), y), \mathtt{s}(\mathtt{b}))\} \to$$
$$\{x =^? \mathtt{p}(\mathtt{s}(\mathtt{a}), y),\ y =^? \mathtt{s}(\mathtt{b})\} \to \{x =^? \mathtt{p}(\mathtt{s}(\mathtt{a}), \mathtt{s}(\mathtt{b})),\ y =^? \mathtt{s}(\mathtt{b})\}$$

which leads to the mgu $[x/\mathtt{p}(\mathtt{s}(\mathtt{a}), \mathtt{s}(\mathtt{b})),\ y/\mathtt{s}(\mathtt{b})]$
(2)

$$\{\mathtt{b}(x, \mathtt{a}, \mathtt{p}(\mathtt{s}(\mathtt{a}))) =^? \mathtt{b}(\mathtt{s}(y), y, \mathtt{p}(x))\} \to \{x =^? \mathtt{s}(y),\ \mathtt{a} =^? y,\ \mathtt{p}(\mathtt{s}(\mathtt{a})) =^? \mathtt{p}(x)\} \to$$
$$\{x =^? \mathtt{s}(y),\ y =^? a,\ \mathtt{p}(\mathtt{s}(\mathtt{a})) =^? \mathtt{p}(\mathtt{s}(y))\} \to \{x = \mathtt{s}(a), y =^? a, \mathtt{p}(\mathtt{s}(a)) =^? \mathtt{p}(\mathtt{s}(a))\}$$

which leads to the mgu $[x/\mathtt{s}(a),\ y/a]$
(3)

$$\{\mathtt{p}(\mathtt{s}(y), \mathtt{s}(x)) =^? \mathtt{p}(x, \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a})))\} \to \{x =^? \mathtt{s}(y),\ \mathtt{s}(x) =^? \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a}))\} \to$$
$$\{x =^? \mathtt{s}(y),\ \mathtt{s}(\mathtt{s}(y)) = \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a}))\} \to \{x =^? \mathtt{s}(y),\ \mathtt{s}(y) =^? \mathtt{p}(\mathtt{a}, \mathtt{a})\}$$

not unifiable since $\mathtt{s} \neq \mathtt{p}$ in $\mathtt{s}(y) =^? \mathtt{p}(\mathtt{a}, \mathtt{a})$.
(4)

$$\{\mathtt{s}(\mathtt{p}(x, z)) =^? \mathtt{s}(\mathtt{p}(\mathtt{p}(\mathtt{s}(\mathtt{a}), y), \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a})))),\ \mathtt{p}(y, z) =^? \mathtt{p}(x, \mathtt{s}(\mathtt{p}(\mathtt{a}, y)))\} \to$$
$$\{\mathtt{p}(x, z) =^? \mathtt{p}(\mathtt{p}(\mathtt{s}(\mathtt{a}), y), \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a}))),\ y =^? x,\ z =^? \mathtt{s}(\mathtt{p}(\mathtt{a}, y))\} \to$$
$$\{x =^? \mathtt{p}(\mathtt{s}(\mathtt{a}), y),\ z =^? \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a})),\ y =^? x,\ z =^? \mathtt{s}(\mathtt{p}(\mathtt{a}, y))\} \to$$
$$\{x =^? \mathtt{p}(\mathtt{s}(\mathtt{a}), x),\ z =^? \mathtt{s}(\mathtt{p}(\mathtt{a}, \mathtt{a})),\ y =^? x,\ z =^? \mathtt{s}(\mathtt{p}(\mathtt{a}, x))\}$$

not unifiable because of $x =^? \mathtt{p}(\mathtt{s}(\mathtt{a}), x)$.

In the following, let $\mathcal{R}$ be a TRS.

A symbol f is **defined** by $\mathcal{R}$ if there is a rule whose left hand-side is headed by f. We will denote with $\mathcal{D}(\mathcal{R})$ the set of all the symbols defined by $\mathcal{R}$.

A **dependency pair** of a rewrite system $\mathcal{R}$ is a pair of terms $(f(\vec{l}), g(\vec{m}))$ where $f(\vec{l}) \to C[g(\vec{m})]$ is a rewrite rule of $\mathcal{R}$ with $g \in \mathcal{D}(\mathcal{R})$.

A **marked dependency pair** of $\mathcal{R}$ is a pair of terms $(f^{\#}(\vec{l}), g^{\#}(\vec{m}))$ such that $(f(\vec{l}), g(\vec{m}))$ is a dependency pair, where $f^{\#}$ is the marked symbol of f.

The **dependency graph** of $\mathcal{R}$ is the directed graph defined as follows:

- its nodes are the set of marked dependency pairs;
- there is an edge from (ℓ_1, r_1) to (ℓ_2, r_2) whenever there are substitutions σ_1 and σ_2 such that $r_1\sigma_1 \to^*_{\mathcal{R}} l_2\sigma_2$.

The **dependency graph approximation** of $\mathcal{R}$ is the directed graph defined as follows:

- its nodes are the set of marked dependency pairs;
- there is an edge from (ℓ_1, r_1) to (ℓ_2, r_2) if $RC(r_1)$ and ℓ_2 are unifiable.

where $RC(r)$ is the term obtained by replacing with a new variable every strict subterm headed by a defined symbol or a variable, in r.

Exercise 2 :
We consider the following TRS:

$$
\begin{aligned}
\mathtt{m}(x, \mathtt{0}) &\to 0 & \mathtt{m}(\mathtt{s}(x), \mathtt{s}(y)) &\to \mathtt{m}(x, y) \\
\mathtt{q}(\mathtt{0}, \mathtt{s}(y)) &\to 0 & \mathtt{q}(\mathtt{s}(x), \mathtt{s}(y)) &\to \mathtt{s}(\mathtt{q}(\mathtt{m}(x, y), \mathtt{s}(y))) \\
\mathtt{p}(\mathtt{0}, y) &\to y & \mathtt{p}(x, \mathtt{s}(y)) &\to \mathtt{s}(\mathtt{p}(x, y)) \\
\mathtt{m}(\mathtt{m}(x, y), z) &\to \mathtt{m}(x, \mathtt{p}(y, z))
\end{aligned}
$$

1. Which rule makes the termination of this TRS not provable with KBO or RPO?
2. What are the defined symbols?
3. Compute the marked dependency pairs.
4. Draw the dependency graph approximation.
5. What are the inequalities that are enough to consider? What can instead be ignored, and why?
6. Find a weakly monotonic polynomial interpretation on integers satisfying those inequalities.

Solution:

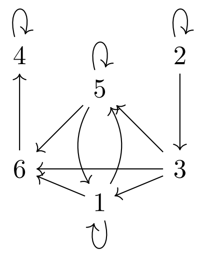
(1) $\mathtt{q}(\mathtt{s}(x), \mathtt{s}(y)) \to \mathtt{s}(\mathtt{q}(\mathtt{m}(x, y), \mathtt{s}(y)))$, since the left-hand side of this rule is embedded in its right-hand side if y is instantiated with $\mathtt{s}(x)$.

(2) The set of defined symbols is $\{\mathtt{m}, \mathtt{q}, \mathtt{p}\}$.

(3) The marked dependency pairs are:

$$
\begin{aligned}
1 &: (\mathtt{m}^{\#}(\mathtt{s}(x), \mathtt{s}(y)),\ \mathtt{m}^{\#}(x, y)) \\
2 &: (\mathtt{q}^{\#}(\mathtt{s}(x), \mathtt{s}(y)),\ \mathtt{q}^{\#}(\mathtt{m}(x, y), \mathtt{s}(y))) \\
3 &: (\mathtt{q}^{\#}(\mathtt{s}(x), \mathtt{s}(y)),\ \mathtt{m}^{\#}(x, y)) \\
4 &: (\mathtt{p}^{\#}(x, \mathtt{s}(y)),\ \mathtt{p}^{\#}(x, y)) \\
5 &: (\mathtt{m}^{\#}(\mathtt{m}(x, y), z),\ \mathtt{m}^{\#}(x, \mathtt{p}(y, z))) \\
6 &: (\mathtt{m}^{\#}(\mathtt{m}(x, y), z),\ \mathtt{p}^{\#}(y, z))
\end{aligned}
$$

(4) Dependency graph approximation:



(5) We need to consider the inequalities:

$$
\begin{aligned}
\mathtt{m}(x,0) &\geq x\\
\mathtt{m}(\mathtt{s}(x),\mathtt{s}(y)) &\geq \mathtt{m}(x,y)\\
\mathtt{q}(0,\mathtt{s}(y)) &\geq 0\\
\mathtt{q}(\mathtt{s}(x),\mathtt{s}(y)) &\geq \mathtt{s}(\mathtt{q}(\mathtt{m}(x,y),\mathtt{s}(y)))\\
\mathtt{p}(0,y) &\geq y\\
\mathtt{p}(x,\mathtt{s}(y)) &\geq \mathtt{s}(\mathtt{p}(x,y))\\
\mathtt{m}(\mathtt{m}(x,y),z) &\geq \mathtt{m}(x,\mathtt{p}(y,z))
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{m}^\#(\mathtt{s}(x),\mathtt{s}(y)) &> \mathtt{m}^\#(x,y)\\
\mathtt{q}^\#(\mathtt{s}(x),\mathtt{s}(y)) &> \mathtt{q}^\#(\mathtt{m}(x,y),\mathtt{s}(y))\\
\mathtt{p}^\#(x,\mathtt{s}(y)) &> \mathtt{p}^\#(x,y)\\
\mathtt{m}^\#(\mathtt{m}(x,y),z) &> \mathtt{m}^\#(x,\mathtt{p}(y,z))
\end{aligned}
$$

Whereas the two inequalities that can be ignored are $\mathtt{q}^\#(\mathtt{s}(x),\mathtt{s}(y)) > \mathtt{m}^\#(x,y)$ and $\mathtt{m}^\#(\mathtt{m}(x,y),z) > \mathtt{p}^\#(y,z)$, that can be ignored since, looking at the dependency graph approximation, they correspond to nodes that don't belongs to any loops. In fact, we just need to consider, for each strongly connected component, the $\geq$ inequalities plus the $>$ inequalities of that strongly connected component.

(6) In this case, it's easy to find a weakly polynomial interpretation over integers that satisfied all the inequalities, instead of considering separately each strongly connected component: $\mathtt{P}_0 = 0$, $\mathtt{P}_\mathtt{s}(X) = X+2$, $\mathtt{P}_\mathtt{m}(X,Y) = X+1$, $\mathtt{P}_\mathtt{q}(X,Y) = 2X$, $\mathtt{P}_{\mathtt{m}^\#}(X,Y) = \mathtt{P}_{\mathtt{q}^\#}(X,Y) = X$, $\mathtt{P}_\mathtt{p}(X,Y) = \mathtt{P}_{\mathtt{p}^\#}(X,Y) = X+Y$.

An **argument filtering** TRS (AFTRS) on the set of functions F is a rewrite system A on $F \cup F'$ for some set of function symbols F' disjoint from F, such that the rules of A are of the form:

- $f(x_1,\ldots,x_n) \to g(y_1,\ldots,y_k)$ with $f \in F$, $g \in F'$, the x_i are pairwise different variables and the $y_j \in \{x_1,\ldots,x_n\}$ are also pairwise different, or
- $f(x_1,\ldots,x_n) \to x_i$ with $f \in F$ and the x_i are pairwise different variables.

Moreover, for every symbol $f \in F$, there is at most one rule of this form.

Exercise 3 :

1. Prove that an AFTRS always terminates and is confluent.
2. Fix A an AFTRS and denote with t_A the normal form of t w.r.t. A. Let IN be a set of inequalities on terms. Prove that if the inequalities

$$\{s_A > t_A \mid s > t \in IN\} \cup \{s_A \geq t_A \mid s \geq t \in IN\}$$

are satisfied by a well-founded weakly monotonic quasi-ordering where both $>$ and $\geq$ are closed under substitution, on terms of $F \uplus F'$, then there is a quasi-ordering $\geq'$ satisfying the inequalities IN.

3. Find a suitable AFTRS that you can use to apply the above result on the set of inequalities obtained in Exercise 2.5 and prove termination using a weakly monotonic polynomial interpretation on integers or a RPO.

Solution:

(1) Termination can be shown, for example, via KBO by defining an order where for each $f \in F$ and $g \in F'$, $w(f) > w(g)$. We can prove strong confluency. Here's a sketch of the proof. Let t be a term and p, p' two positions where we can apply respectively the rewriting rules R and R'. If $p = p'$ then $R = R'$ from the conditions of AFTRS and the result holds. If $p \not< p'$ and $p' \not< p$ then trivially $t \xrightarrow{R}\xrightarrow{R'} t'$ and $t \xrightarrow{R'}\xrightarrow{R} t'$. Let now R be of the form $f(x_1, \ldots, x_n) \to g(y_1, \ldots, y_k)$ and let R' be of the form $f'(x'_1, \ldots, x'_m) \to g'(y'_1, \ldots, y'_j)$. W.l.o.g. suppose $p' < p$. Then there exists i such that $p' \leq ip$. By applying R to $t|_p$ one of the following will holds

- if there exists l such that, $x_i = y_l$, then R' can still be applied and it will hold that $t \xrightarrow{R}\xrightarrow{R'} t'$ and $t \xrightarrow{R'}\xrightarrow{R} t'$;
- otherwise, if x_i does not appear in the right side of R, then R' can not longer be applied but it will hold that $t \xrightarrow{R} t'$ and $t \xrightarrow{R'}\xrightarrow{R} t'$.

(2) Assuming that the normalized inequalities are satisfied by $\geq$, a relation $\geq'$ on terms is defined where the terms are first normalized and then compared w.r.t. $\geq$, i.e. $s \geq' t$ if and only if $s_A \geq t_A$. It is straightforward to see that $\geq'$ is a well-founded quasi-ordering satisfying the inequalities IN. For any substitution σ, let σ_A denote the substitution which results from σ by normalizing all terms in its range, w.r.t. A. Then, for all terms t and all substitutions σ we have $(t\sigma)_A = t_A\sigma_A$. Hence, both $\geq'$ and $>'$ are closed under substitution. Moreover, $\geq'$ (and therefore $>'$) is weakly monotonic, because $s_A \geq t_A$ implies $f(\ldots, s_A, \ldots)_A \geq f(\ldots, t_A, \ldots)_A$ which is equivalent to $f(\ldots, s, \ldots)_A \geq f(\ldots, t, \ldots)_A$.

(3) We define the AFTRS $\{\mathtt{m}(x, y) \to M(x),\ \mathtt{p}(x, y) \to P(y)\}$. After normalizing all inequalities of Exercise 1.5 we have:

$$\begin{aligned}
M(x) &\geq x \\
M(\mathtt{s}(x)) &\geq M(x) \\
\mathtt{q}(0, \mathtt{s}(y)) &\geq 0 \\
\mathtt{q}(\mathtt{s}(x), \mathtt{s}(y)) &\geq \mathtt{s}(\mathtt{q}(M(x), \mathtt{s}(y))) \\
P(y) &\geq y \\
P(\mathtt{s}(y)) &\geq \mathtt{s}(P(y)) \\
M(M(x)) &\geq M(x)
\end{aligned}$$

$$\begin{aligned}
\mathtt{m}^{\#}(\mathtt{s}(x), \mathtt{s}(y)) &> \mathtt{m}^{\#}(x, y) \\
\mathtt{q}^{\#}(\mathtt{s}(x), \mathtt{s}(y)) &> \mathtt{q}^{\#}(M(x), \mathtt{s}(y)) \\
\mathtt{p}^{\#}(\mathtt{s}(x), y) &> \mathtt{p}^{\#}(x, y) \\
\mathtt{m}^{\#}(M(x), z) &> \mathtt{m}^{\#}(x, P(z))
\end{aligned}$$

That can be proved terminating using RPO with lexicographic ordering w.r.t. the order (other orders are also possible):

$$\mathtt{m}^{\#} > q^{\#} > q > p^{\#} > P > s > M$$

Otherwise, a weakly monotonic polynomial interpretation on integers can be defined as follows: $P_{\mathtt{0}} = 0$, $P_P(X) = X$, $P_M(X) = X + 1$, $P_{\mathtt{q}}(X, Y) = X^2 + Y$, $P_{\mathtt{s}}(X) = X + 2$, $P_{\mathtt{p}^{\#}}(X, Y) = P_{\mathtt{q}^{\#}}(X, Y) = P_{\mathtt{m}^{\#}}(X, Y) = X + Y$.

We say that a term is a **minimal non-terminating term** if all its proper subterms are terminating but he is not.

Let C be a cycle in the dependency graph of $\mathcal{R}$ such that every dependency pair symbol in C has positive arity. A **simple projection** for C is a mapping π that assign to every n-ary marked symbol $f^\#$ in C an argument position $i \in [1,n]$. We define $\pi(f^\#(t_1,\dots,t_n)) = t_{\pi(f^\#)}$, where $f^\#(t_1,\dots,t_n)$ is a term and $f^\#$ marked symbol in C.

Theorem. For every non-terminating TRS $\mathcal{R}$ there exists a cycle C in the dependency graph of $\mathcal{R}$ and an infinite rewrite sequence in $\mathcal{R} \cup C$ of the form

$$t_1 \to_{\mathcal{R}}^* t_2 \to_C t_3 \to_{\mathcal{R}}^* t_4 \to_C t_5 \to_{\mathcal{R}}^* \cdots$$

where $t_1 = f^\#(s_1,\dots,s_n)$ is headed by a marked symbol, $f(s_1,\dots,s_n)$ is a minimal non-terminating term and all rules of C are applied infinitely often.

Exercise 4 :

1. Let S be a rewrite system and such that each defined symbol has positive arity. Prove that if every cycle C of the dependency graph of S has a simple projection π such that $\pi(C) \subseteq \trianglerighteq$ and $\pi(C) \cap \triangleright \neq \emptyset$, where $\pi(C) = \{(\pi(s),\pi(t)) \mid (s,t) \in C\}$ and $\trianglerighteq$ is the subterm relation, then S terminates.

Consider the following rewriting system:

$$\begin{aligned}
\mathtt{m}(\mathtt{1}) &\to \mathtt{1} & \mathtt{m}(\mathtt{a}(x,y)) &\to \mathtt{a}(\mathtt{s}(x),\mathtt{m}(y)) \\
\mathtt{q}(\mathtt{0},\mathtt{0}) &\to \mathtt{a}(\mathtt{0},\mathtt{1}) & \mathtt{q}(\mathtt{s}(x),\mathtt{0}) &\to \mathtt{1} \\
\mathtt{q}(\mathtt{s}(x),\mathtt{s}(y)) &\to \mathtt{m}(\mathtt{q}(x,y)) & \mathtt{q}(\mathtt{0},\mathtt{s}(y)) &\to \mathtt{a}(\mathtt{0},\mathtt{q}(\mathtt{s}(\mathtt{0}),\mathtt{s}(y)))
\end{aligned}$$

2. Compute the marked dependency pairs and the dependency graph approximation.
3. Prove the termination of the rewrite system by finding a suitable simple projection that satisfied the constraints in question 1.

Solution:

(1) From the Theorem, suppose to the contrary that there exists a rewrite sequence

$$t_1 \to_{\mathcal{R}}^* u_1 \to_C t_2 \to_{\mathcal{R}}^* u_2 \to_C t_3 \to_{\mathcal{R}}^* \cdots$$

where $t_1 = f^\#(s_1,\dots,s_n)$ is headed by a marked symbol, $f(s_1,\dots,s_n)$ is a minimal non-terminating term and all rules of C are applied infinitely often. We apply the simple projection to this rewrite sequence:

- Consider $u_i \to_C t_{i+1}$. There exists a dependency pair $l \to r \in C$ and a substitution σ such that $u_i = l\sigma$ and $t_{i+1} = r\sigma$. We have $\pi(u_i) = \pi(l)\sigma$ and $\pi(t_{i+1}) = \pi(r)\sigma$. Since $\pi(l) \to \pi(r) \in \pi(C)$, by hypothesis it holds $\pi(l) \trianglerighteq \pi(r)$. So $\pi(l) = \pi(r)$ or $\pi(l) \triangleright \pi(r)$. In the former case, trivially $\pi(u_1) = \pi(t_{i+1})$. In the latter case, the closure under substitution of $\triangleright$ yields $\pi(u_i) \triangleright \pi(t_{i+1})$. Because of the assumption $\pi(C) \cap \triangleright \neq \emptyset$, and all rules of C are applied infinitely often, $\pi(u_i) \triangleright \pi(t_{i+1})$ will hold for infinitely many i.
- Consider now $t_i \to_{\mathcal{R}}^* u_i$. All steps in this sequence take place below the (marked) root symbol, which is therefore the same for t_i and u_i. Therefore $\pi(t_i) \to_{\mathcal{R}}^* \pi(u_i)$ holds.

By applying our simple projection π to the rewrite sequence, we transform it into a infinite $\to_{\mathcal{R}} \cup \triangleright$ sequence containing infinitely many $\triangleright$ steps, starting from $\pi(t_1)$. Since $\triangleright$ is well-founded, the sequence must also contain infinitely many $\to_{\mathcal{R}}$ steps. By making repeated use of the commutation $(\triangleright \to_{\mathcal{R}}) \subseteq (\to_{\mathcal{R}} \triangleright)$ we obtain an infinite sequence of $\to_{\mathcal{R}}$ starting from $\pi(t_1)$. Therefore $\pi(t_1)$ is not terminating w.r.t. $\mathcal{R}$. But $f(s_1,\dots,s_n) \triangleright \pi(t_1)$ and $f(s_1,\dots,s_n)$ is a minimal non-terminating term: contradiction.

(2) The defined symbols are $\{\mathtt{m},\mathtt{q}\}$. The marked dependency pairs are:

$$\begin{aligned}
&1 : (\mathtt{m}^\#(\mathtt{a}(x,y)),\ \mathtt{m}^\#(y)) \\
&2 : (\mathtt{q}^\#(\mathtt{s}(x),\mathtt{s}(y)),\ \mathtt{m}^\#(\mathtt{q}(x,y))) \\
&3 : (\mathtt{q}^\#(\mathtt{s}(x),\mathtt{s}(y)),\ \mathtt{q}^\#(x,y)) \\
&4 : (\mathtt{q}^\#(\mathtt{0},\mathtt{s}(y)),\ \mathtt{q}^\#(\mathtt{s}(\mathtt{0}),\mathtt{s}(y)))
\end{aligned}$$

whereas the dependency graph approximation is

$$\begin{tikzcd}
1 \arrow[loop left] & 2 \arrow[l] \\
3 \arrow[loop left] \arrow[ur] \arrow[r, bend right] & 4 \arrow[u] \arrow[l, bend right]
\end{tikzcd}$$

There are 3 loops: $\{1\}$, $\{3\}$, $\{3,4\}$. We define $\pi(f^{\#}) = 1$ and $\pi(g^{\#}) = 2$. For the first loop it holds $\mathtt{a}(x, y) \rhd y$; for the second loop $\mathtt{s}(y) \rhd y$ and for the last loop $\mathtt{s}(y) \rhd y$ and $\mathtt{s}(y) \unrhd \mathtt{s}(y)$. The conditions to apply the result in question 1 are therefore satisfied and the TRS terminates.