Rewriting Techniques: TD 2

23-11-2017

Exercise 1 :
Let A and B be two terminating relations. Show that if $AB \subseteq BA$ then $A \cup B$ terminates using well-founded induction.

> **Solution:**
>
> We prove that $(A \cup B)^* \subseteq B^*A^*$ by well-founded induction on steps of $(A \cup B)^*$. The induction hypothesis is: $t(\rightarrow_{A\cup B})^n t' \implies t \rightarrow_A^* \rightarrow_B^* t'$. W.l.o.g. suppose $t \rightarrow_{A\cup B}^{n+1} t'$ such that $t \rightarrow_A t'' \rightarrow_B (\rightarrow_{A\cup B})^{n-1} t'$ for some $n \in \mathbb{N}$. Then by hypothesis $AB \subseteq BA$ it holds that there exists t''' such that $t \rightarrow_B t''' \rightarrow_A (\rightarrow_{A\cup B})^{n-1} t'$. By inductive hypothesis, since $t'''(\rightarrow_{A\cup B})^n t'$, $t''' \rightarrow_B^* \rightarrow_A^* t'$ and therefore $t \rightarrow_B^* \rightarrow_A^* t'$. Since A and B are terminating relations, B^*A^* is a terminating relation. It follows that $A \cup B$ is terminating.

Given a strict order $>$ on a set A, we define the corresponding **multiset order** $>_{\mathrm{mul}}$ on $\texttt{Mult}(A)$ as follows: $M >_{\mathrm{mul}} N$ if and only if there exist $X, Y \in \texttt{Mult}(A)$ such that

1. $\emptyset \neq X \subseteq M$;
2. $N = (M \setminus X) \cup Y$;
3. $\forall y \in Y\ \exists x \in X\ x > y$.

Exercise 2 :
Prove that $M >_{\mathrm{mul}} N$ if and only if $M \neq N$ and for all $n \in N \setminus M$ there exists $m \in M \setminus N$ such that $m > n$.

> **Solution:**
>
> ($\Rightarrow$) Assume $M >_{\mathrm{mul}} N$, X and Y as in the definition. $M \neq N$ follows from irreflexivity of $>_{\mathrm{mul}}$. For the second conjunct, suppose $y_1 \in N \setminus M = ((M \setminus X) \cup Y) \setminus M = ((M \cup Y) \setminus X) \setminus M = ((M \cup Y) \setminus M) \setminus X = Y \setminus X$, where the second equality holds because $X \subseteq M$. Hence there is a $y_2 \in X$ such that $y_2 > y_1$. Either $y_2 \in X \setminus Y = (M \setminus (M \setminus X)) \setminus Y = M \setminus ((M \setminus X) \cup Y) = M \setminus N$, in which case we are done, or $y_2 \in X \cap Y$ (where $X \cap Y)(x) = \min(X(x), Y(x)))$, in which case there is $y_3 \in X$ such that $y_3 > y_2$. Because or multisets are finite and $>$ is a strict order, there is no infinite ascending chain $y_1 < y_2 < \ldots$ in $X \cap Y$. This process therefore terminate with some $y_n \in X \setminus Y = M \setminus N$. Transitivity yields $y_n > y_1$.
>
> ($\Leftarrow$) Let $N \setminus M = Y$ and $M \setminus N = X$. Since $M \neq N$, it cannot be that $M = N = \emptyset$. Moreover it holds that $X \neq \emptyset$. Indeed, suppose that $M \setminus N$ is empty, then for all e $M(e) \leq N(e)$ and since $M \neq N$ it will hold that $N \neq \emptyset$, which will lead $N \setminus M$ not empty and therefore (contradiction), for the existential in the second condition X cannot be empty. Moreover from it's definition $X \subseteq M$. Moreover $N = (M \setminus (M \setminus N)) \cup (N \setminus M)$ and so $N = (M \setminus X) \cup Y$. From the last hypothesis it holds that for all $n \in Y$ there exists $m \in X$ such that $m > n$. We conclude $M >_{\mathrm{mul}} N$.

The **lexicographic order** $>_{\mathrm{lex}}$ for the Cartesian product $\times$ of two domains $(A, >_A)$ and $(B, >_B)$ is defined as follows: $(a_1, b_1) >_{\mathrm{lex}} (a_2, b_2)$ if and only if $a_1 > a_2$ or $a_1 = a_2$ and $b_1 > b_2$. This order can be readily extended con Cartesian products of arbitrary length by recursively applying this definition, i.e by observing that $A \times B \times C = A \times (B \times C)$.

In the following, let Σ be a finite signature, V a set of variables and $T(\Sigma, V)$ the terms built from those sets.

Let $\mathtt{status}(f) \in \{\mathrm{mul}, \mathrm{lex}\}$ a *status* function on Σ and let $>$ be a strict order on Σ. The **recursive path order** $>_{\mathrm{rpo}}$ on $T(\Sigma, V)$ induced by $>$ is defined as follows. $s >_{\mathrm{rpo}} t$ if and only if one of the following holds:

1. t is a variable appearing in s and $s \neq t$, or

let $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$,

2. there exists $i \in [1, m]$ such that $s_i \geq_{\mathrm{rpo}} t$, or
3. $f > g$ and $s >_{\mathrm{rpo}} t_j$ for all $j \in [1, n]$, or
4. $f = g$, for all $j \in [1, n]$ it holds $s >_{\mathrm{rpo}} t_j$ and $(s_1, \ldots, s_m)(>_{\mathrm{rpo}})_{\mathtt{status}(f)}(t_1, \ldots, t_m)$.

The **lexicographic path order** is a recursive path order s.t. for all $f \in \Sigma$, $\mathtt{status}(f) = \mathrm{lex}$, whereas the **multiset path order** is a recursive path order s.t. for all $f \in \Sigma$, $\mathtt{status}(f) = \mathrm{mul}$, where we define $(s_1, \ldots, s_m)(>_{\mathrm{rpo}})_{\mathrm{mul}}(t_1, \ldots, t_m)$ as $\{\!\!\{ s_1, \ldots, s_m \}\!\!\}(>_{\mathrm{rpo}})_{\mathrm{mul}}\{\!\!\{ t_1, \ldots, t_m \}\!\!\}$.

Exercise 3 :

1. Prove the termination of the Ackermann's function by well-founded induction.

$$\begin{aligned}
\mathtt{Ack}\ 0\ y &= y + 1 \\
\mathtt{Ack}\ x\ 0 &= \mathtt{Ack}\ (x-1)\ 1 \\
\mathtt{Ack}\ x\ y &= \mathtt{Ack}\ (x-1)\ (\mathtt{Ack}\ x\ (y-1))
\end{aligned}$$

The following rewrite system simulates this function.

$$\begin{gathered}
\mathtt{a}(\mathtt{0}, y) \to \mathtt{s}(y) \\
\mathtt{a}(\mathtt{s}(x), \mathtt{0}) \to \mathtt{a}(x, \mathtt{s}(\mathtt{0})) \\
\mathtt{a}(\mathtt{s}(x), \mathtt{s}(y)) \to \mathtt{a}(x, \mathtt{a}(\mathtt{s}(x), y))
\end{gathered}$$

2. Consider the well-founded domain $(\mathtt{Mult}(\mathbb{N} \times \mathbb{N}), (>_{\mathrm{lex}})_{\mathrm{mul}})$. Prove the termination using the following abstraction:

$$\begin{aligned}
\phi:\ T(\{\mathtt{a}, \mathtt{s}\}, X) &\to \mathtt{Mult}(\mathbb{N} \times \mathbb{N}) \\
t &\to \{\!\!\{ (|u|, |v|) \mid t|_{p \in \mathrm{Pos}(t)} = \mathtt{a}(u, v) \}\!\!\}
\end{aligned}$$

   where $|\mathtt{0}| = 1$, $|\mathtt{a}(x, y)| = |x| + |y| + 1$ and $|\mathtt{s}(x)| = |x| + 1$.
3. Prove the termination using a RPO.

Solution:

(1) Induction on $(\mathbb{N} \times \mathbb{N}, >_{\mathrm{lex}})$. We prove that the calculus of $\mathtt{Ack}\ u\ v$ terminates by induction (u, v) ordered lexicographically on integers.

- Base cases: $\mathtt{Ack}$ terminates for $(0, n)$, $n \in \mathbb{N}$;
- We need to show that $\mathtt{Ack}$ terminates for (n, m), $n > 0$. Induction hypothesis: $\mathtt{Ack}$ terminates for all (j, k) such that $j < n$ or $(j = n$ and $k < m)$. If $m = 0$ then by induction hypothesis the function terminates since $(n-1, 1) < (n, m)$. Instead, if $m > 0$, by induction hypothesis the function terminates on input $(n, m-1)$ with output r and terminates on input $(n-1, r)$.

(2) Let $s \to t$,

- if $s = C[\mathtt{a}(0, t')]$ and $t = C[\mathtt{s}(t')]$, then $\phi(t) = \phi(s) \setminus \{\!\!\{ (1, |t'|) \}\!\!\}$. Therefore it holds $\phi(s)(>_{\mathrm{lex}})_{\mathrm{mul}}\phi(t)$;
- if $s = C[\mathtt{a}(\mathtt{s}(t'), 0)]$ and $t = C[\mathtt{a}(\mathtt{s}(t'), \mathtt{s}(0))]$, then $\phi(t) = \phi(s) \cup \{\!\!\{ (|t'|, 2) \}\!\!\} \setminus \{\!\!\{ (|t'| + 1, 1) \}\!\!\}$;

- if $s = C[\mathtt{a}(\mathtt{s}(t'), \mathtt{s}(t''))]$ and $t = C[\mathtt{a}(t', \mathtt{a}(\mathtt{s}(t'), t''))]$, then

$$\phi(t) = \phi(s) \cup \{\!\!|\ (|t'|, |t'| + |t''| + 2), (|t'| + 1, |t''|)\ |\!\!\} \setminus \{\!\!|\ (|t'| + 1, |t''| + 1)\ |\!\!\}.$$

(3) Let $>$ be such that $\mathtt{a} > \mathtt{s}$ and let $\mathtt{status}(\mathtt{a}) = \mathtt{status}(\mathtt{s}) = \text{lex}$. It holds:

- $\mathtt{a}(0, t) >_{\text{rpo}} \mathtt{s}(t)$, since $\mathtt{a} > \mathtt{s}$ and $\mathtt{a}(0, t) >_{\text{rpo}} t$;
- $\mathtt{a}(\mathtt{s}(t), 0) >_{\text{rpo}} \mathtt{a}(t, \mathtt{s}(0))$ since $\mathtt{a}(\mathtt{s}(t), 0) > t$, $\mathtt{a}(\mathtt{s}(t), 0) > \mathtt{s}(0)$ and $(\mathtt{s}(t), 0)(>_{\text{rpo}})_{\text{lex}}(t, \mathtt{s}(0))$;
- $\mathtt{a}(\mathtt{s}(t), \mathtt{s}(t')) >_{\text{rpo}} \mathtt{a}(t, \mathtt{a}(\mathtt{s}(t), t'))$ since

$$\mathtt{a}(\mathtt{s}(t), \mathtt{s}(t')) >_{\text{rpo}} t,\ \ \mathtt{a}(\mathtt{s}(t), \mathtt{s}(t')) >_{\text{rpo}} \mathtt{a}(\mathtt{s}(t), t'),\ \ (\mathtt{s}(t), \mathtt{s}(t'))(>_{\text{rpo}})_{\text{lex}}(t, \mathtt{a}(\mathtt{s}(t), t'))$$

Exercise 4 :
Show that is not possible to prove termination using lexicographic path ordering of the following term rewrite system:

$$\{\ \mathtt{a}(\mathtt{a}(x)) \to \mathtt{s}(x),\ \ \mathtt{s}(\mathtt{s}(x)) \to \mathtt{a}(x)\ \}$$

Solution:

Suppose $\mathtt{a} > \mathtt{s}$. Let $s \to t$ with $s = C[\mathtt{s}(\mathtt{s}(t')]$ and $t = C[\mathtt{a}(t')]$. Then we need to prove that $\mathtt{s}(\mathtt{s}(t')) >_{\text{rpo}} \mathtt{a}(t')$, which does not hold. If instead $\mathtt{s} > \mathtt{a}$, then from the first rule we get $\mathtt{a}(\mathtt{a}(t')) >_{\text{rpo}} \mathtt{s}(t')$ which, again, does not hold.

Exercise 5 :
Show that the termination of the following rewriting system cannot be proven with lexicographic path order but can be proven with multiset path order.

$$0 + x \to 0 \qquad\qquad 0 \times x \to x$$
$$\mathtt{s}(x) + y \to \mathtt{s}(x + y) \qquad\qquad \mathtt{s}(x) \times y \to (y \times x) + y$$

Solution:

We need to impose the precedence $\times > + > \mathtt{s}$. With this, the lexicographic path order cannot orient the last rule as it does not hold that $(\mathtt{s}(x), y) >_{\text{lex}} (y, x)$. Instead, it holds $\{\!\!|\ \mathtt{s}(x), y\ |\!\!\} >_{\text{mul}} \{\!\!|\ y, x\ |\!\!\}$ since $\mathtt{s}(x)$ dominates x.

Let $>$ be a strict order on Σ and $w : \Sigma \cup V \to \mathbb{R}_0^+$ be a weight function $w : \Sigma \cup V \to \mathbb{R}_0^+$. The **Knuth-Bendix order** (KBO) $>_{\text{kbo}}$ on $T(\Sigma, V)$ induced by $>$ and w is defined as follows: for $s, t \in T(\Sigma, V)$ we have $s >_{\text{kbo}} t$ if and only if $|s|_x \geq |t|_x$ for all $x \in V$ and $w(s) \geq w(t)$. Moreover, if $w(s) = w(t)$ then one of the following properties must hold:

1. There are a unary function f, $x \in V$ and $n \in \mathbb{N}^{\geq 1}$ s.t. $s = f^n(x)$ and $t = x$, or
2. there exist function symbols f, g s.t. $f > g$ and $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$, or
3. there exist a function symbol f such that $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$ and

$$(s_1, \ldots, s_m)(>_{\text{kbo}})_{\text{lex}}(t_1, \ldots, t_m).$$

A weight function $w : \Sigma \cup V \to \mathbb{R}_0^+$ is called **admissible** if and only if it satisfy the following properties w.r.t. a strict order $>$:

1. There exists $w_0 \in \mathbb{R}_0^+ \setminus \{0\}$ s.t. $w(x) = w_0$ for all $x \in V$ and $w(c) \geq w_0$ for all constants $c \in \Sigma$.
2. If $f \in \Sigma$ is a unary function symbol of weight $w(f) = 0$ then f is the greatest element in Σ, i.e. $f \geq g$ for all $g \in \Sigma$.

Exercise 6 :
Using a KBO, prove the termination of:

1. $\{\mathtt{l}(x) + (y + z) \to x + (\mathtt{l}(\mathtt{l}(y)) + z),\ \mathtt{l}(x) + (y + (z + w)) \to x + (z + (y + w))\ \}$
2. $\{\ \mathtt{r}^n(\mathtt{l}^k(x)) \to \mathtt{l}^k(\mathtt{r}^m(x))\ \}$, where $n, k > 0$ and $m \geq 0$.

Solution:

(1) Let $\mathtt{l} > +$, $w(s) = 0$, $w(+) = w(x) > 0$ for each variable x. For both rules, it holds that the weight does not change after the rewrite step. To prove that $\mathtt{l}(x)+(y+z) >_{\text{kbo}} x+(\mathtt{l}(\mathtt{l}(y))+z)$ we therefore need to prove the third condition, which holds since $\mathtt{l}(x) >_{\text{kbo}} x$. Similarly, to prove $\mathtt{l}(x) + (y + (z + w)) >_{\text{kbo}} x + (z + (y + w))$, it is sufficient to show that $\mathtt{l}(x) >_{\text{kbo}} x$. Notice that it does not holds $(y + (z + w) >_{\text{kbo}} z + (y + w)$ because of the ordering of the variables.

(2) Let $\mathtt{r} > \mathtt{l}$, $w(\mathtt{r}) = 0$ and $w(\mathtt{l}) = 1$. It holds that w is admissible. Let $s \to t$ with $s = C[\mathtt{r}^n(\mathtt{l}^k(t'))]$ and $t = C[\mathtt{l}^k(\mathtt{r}^m(t'))]$. From the definition of w, it holds that $w(s) = w(t)$ since the number of occurrences of the function symbol $\mathtt{l}$ is the same in s and t. By applying the definition of $>_{\text{kbo}}$, we get that we need to show $\mathtt{r}^n(\mathtt{l}^k(t')) >_{\text{kbo}} \mathtt{l}^k(\mathtt{r}^m(t'))$, which holds since $\mathtt{r} > \mathtt{l}$.

A strict order $>$ on $T(\Sigma, V)$ is called a **rewrite order** if and only iff

1. is *compatible*: for all $s_1, s_2 \in T(\Sigma, V)$, all $f \in \Sigma$, if $s_1 > s_2$ then
$$f(t_1, \ldots, t_{i-1}, s_1, t_{i+1}, \ldots, t_n) > f(t_1, \ldots, t_{i-1}, s_2, t_{i+1}, \ldots, t_n)$$
where n is the arity of f;
2. is *closed under substitution*: for all $s_1, s_2 \in T(\Sigma, V)$ and all substitutions $\sigma : V \to T(\Sigma, V)$, if $s_1 > s_2$ then $\sigma(s_1) > \sigma(s_2)$.

A strict order $>$ on $T(\Sigma, V)$ satisfies the **subterm property** (and is called *simplification order*) if and only if it is a rewrite order such that for all terms $t \in T(\Sigma, V)$ and all positions $p \in \text{Pos}(t) \setminus \{\epsilon\}$ it holds $t > t|_p$.

Exercise 7 :
In the following, we refer to s, t and w as in the definition of KBO. We will now prove some properties of this order to make it more clear.

1. Assume that f is of arity 1, $w(f) = 0$ and that there is g such that $f \not> g$. Prove that under this conditions $>_{\text{kbo}}$ does not satisfy the subterm property.

Prove that, if w is admissible for the strict order $>$ then $>_{\text{kbo}}$ on $T(\Sigma, V)$ induced by $>$ and w has the subterm property. To do so, prove the followings:

2. Assume that $w(s) = w(t)$ and that t is a strict subterm of s. Prove that there exist a unary function f and a positive integer k such that $w(f) = 0$ and $s = f^k(t)$;
3. Prove that $>_{\text{kbo}}$ is a strict order;
4. Prove that $>_{\text{kbo}}$ is a rewrite order;
5. Conclude that $>_{\text{kbo}}$ has the subterm property.

Solution:

(1) Let $t = g(t_1, \ldots, t_n)$ be an arbitrary term with root symbol g, define $s = f(t)$ and let $s \to t$. Since $w(f) = 0$ we have $w(s) = w(t)$. Obviously, the first and third condition of KBO cannot hold, so $>_{\text{kbo}}$ holds if and only if the second condition holds. But this cannot happen since $f \not> g$.

(2) Proof by induction on the size of s. Since t is a strict subterm of s, there are an $n \geq 1$ and an n-ary function symbol f such that $s = f(s_1, \ldots, s_n)$ and t is a subterm of s_i for some $i \in [1, n]$. First we show that $n = 1$ and $w(f) = 0$:

- Assume $n > 1$. We have $w(s) = w(f) + \sum_{j=1}^{n} w(s_j)$ and (since w admissible) we know that for all j, $w(s_j) \geq w_0 > 0$. Thus, $n > 1$ implies $w(s) > w(s_i)$ and therefore $w(s) > w(t)$. This contradicts the hypothesis $w(s) = w(t)$.
- Assume $w(f) > 0$, then, even for $n = 1$, $w(s) = w(f) + \sum_{j=1}^{n} w(s_j) > w(s_i) \geq w(t)$. This contradicts the hypothesis $w(s) = w(t)$.

This shows that $s = f(s')$ where f is unary, $w(f) = 0$ and s' has t as a subterm. For $s' = t$ we are done. Otherwise we can apply the induction hypothesis since t is a strict subterm of s', $w(s') = w(s) = w(t)$ and $|s'| < |s|$.

(3) Assume that $>_{\mathrm{kbo}}$ is not *irreflexive*. then s be a term of minimal size such that $s >_{\mathrm{kbo}} s$. Since $w(s) = w(s)$ and the root symbol is the same, then we obtain $s_i >_{\mathrm{kbo}} s_i$ for all $i \in [1, n]$ where n is the arity of the root symbol of s. This contradicts the minimality of s. To show *transitivity* assume $r >_{\mathrm{kbo}} s$ and $s >_{\mathrm{kbo}} t$, we prove $r >_{\mathrm{kbo}} t$ by induction on the size of r.

- From $r >_{\mathrm{kbo}} s$ and $s >_{\mathrm{kbo}} t$ we deduce that, for all variables x, $|r|_x \geq |s|_x$ and $|s|_x \geq |t|_x$ hold, thus e have $|r|_x \geq |t|_x$. The variables condition is therefore satisfied.
- $r >_{\mathrm{kbo}} s$ and $s >_{\mathrm{kbo}} t$ also yield $w(r) \geq w(s)$ and $w(s) \geq w(t)$, which implies $w(r) \geq w(t)$. Moreover if $w(r) > w(s)$ or $w(s) > w(t)$ then $w(r) > w(t)$ and we are done.

We can assume $w(r) = w(s) = w(t)$. Moreover the second point of the definition cannot hold for $r >_{\mathrm{kbo}} s$, since $s >_{\mathrm{kbo}} t$ implies that s is not a variable. Therefore r and s have a function symbol as root, i.e. $r = f(r_1, \ldots, r_l)$ and $s = g(s_1, \ldots, s_m)$, such that $f \geq g$.

1. If $s >_{\mathrm{kbo}} t$ satisfies the first condition then $t = x$ for a variable x and $|r|_x \geq |t|_x$ implies that x occurs in r. Since the root symbol of r is a function symbol we have $r \neq x$ and from the previous point we have $r >_{\mathrm{kbo}} t$.
2. If instead $s >_{\mathrm{kbo}} t$ satisfies the second or third condition, then we know that there exists a function symbol h such that $g \geq h$ and $t = h(t_1, \ldots, t_n)$. If $f > g$ or $g > h$ then we have $f > h$ and by the second condition $r >_{\mathrm{kbo}} t$. Otherwise, assume $f = g = h$. Then both $r >_{\mathrm{kbo}} s$ and $s >_{\mathrm{kbo}} t$ satisfy the third condition. By induction hypothesis, from the definition of $(>_{\mathrm{kbo}})_{\mathrm{lex}}$ we get $r >_{\mathrm{kbo}} t$.

(4) We first show that $>_{\mathrm{kbo}}$ is compatible. Assume $s_1 >_{\mathrm{kbo}} s_2$ and f n-ary function symbol. We must show that the following holds

$$f(t_1, \ldots, t_{i-1}, s_1, t_i, \ldots, t_n) >_{\mathrm{kbo}} f(t_1, \ldots, t_{i-1}, s_2, t_i, \ldots, t_n)$$

From $s_1 >_{\mathrm{kbo}} s_2$ we can deduce that $|s_1|_x \geq |s_2|_x$ for all variables x. This obviously implies

$$|f(t_1, \ldots, t_{i-1}, s_1, t_i, \ldots, t_n)|_x \geq |f(t_1, \ldots, t_{i-1}, s_2, t_i, \ldots, t_n)|_x$$

Moreover, if $w(s_1) > w(s_2)$ then

$$w(f(t_1, \ldots, t_{i-1}, s_1, t_i, \ldots, t_n)) > w(f(t_1, \ldots, t_{i-1}, s_2, t_i, \ldots, t_n))$$

and yields our thesis. Assume instead $w(s_1) = w(s_2)$. This implies that

$$w(f(t_1, \ldots, t_{i-1}, s_1, t_i, \ldots, t_n)) = w(f(t_1, \ldots, t_{i-1}, s_2, t_i, \ldots, t_n))$$

and since the root symbols of the two terms are the same, the thesis holds if and only if the third condition of KBO is satisfied. This is trivial since $t_1 = t_1$,$\ldots$,$t_{i-1} = t_{i-1}$ and $s_1 >_{\mathrm{kbo}} s_2$. To show instead that $>_{\mathrm{kbo}}$ is closed under substitution, assume $s_1 >_{\mathrm{kbo}} s_2$ and let $\sigma : V \to T(\Sigma, V)$ be a substitution. We show $\sigma(s_1) >_{\mathrm{kbo}} \sigma(s_2)$ by induction on the size of s_1. First, consider the variable condition. Let X be the set of variables appearing in s_1. Because of $s_1 >_{\mathrm{kbo}} s_2$ we know that $|s_1|_x \leq |s_2|_x$ for all variables x. For an arbitrary variable x we have

$$|\sigma(s_1)|_x - |\sigma(s_2)|_x = \sum_{y \in X} |\sigma(y)|_x (|s_1|_y - |s_2|_y) \geq 0$$

Thus the variable condition is satisfied. A similar computation can be done for weights:

$$w(\sigma(s_1)) - w(\sigma(s_2)) = w(s_1) - w(s_2) + \sum_{y \in X} (|s_1|_y - |s_2|_y)(w(\sigma(y)) - w_0)$$

For all $y \in X$, it holds $|s_1|_y - |s_2|_y \geq 0$ and $w(\sigma(y)) - w_0 \geq 0$. Consequently $w(s_1) > w(s_2)$ implies $w(\sigma(s_1)) > w(\sigma(s_2))$ which yields $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$. Assume instead that $w(s_1) = w(s_2)$ and hence $w(\sigma(s_1)) \geq w(\sigma(s_2))$.If $w(\sigma(s_1)) > w(\sigma(s_2))$, then $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$. Otherwise we consider one subcase for each condition of KBO:

1. If $s_1 >_{\text{kbo}} s_2$ holds for the first condition, then $s_1 = f^k(x)$ and $s_2 = x$ for a unary symbol f of weight 0 a variable x and a positive integer k. We show $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$ by induction on the size of $\sigma(x)$. If $\sigma(x) = y$ is a variable, then the result trivially holds for the first condition of KBO. Otherwise $\sigma(x) = g(t_1, \ldots, t_n)$ for a function symbol g of arity n. If $f \neq g$ then $f > g$ from the admissibility of w and thus $\sigma(s_1) = f^k(g(t_1, \ldots, t_n)) >_{\text{kbo}} g(t_1, \ldots, t_n) = \sigma(s_2)$ holds from the second condition of KBO. If $f = g$ then the third condition must apply and we need to prove that $f^k(t_1) >_{\text{kbo}} t_1$. By taking a substitution σ' such that $\sigma'(x) = t_1$ then the induction hypothesis (appliable since $\sigma'(x)$ is smaller than $\sigma(x)$) yields $f^k(t_1) = \sigma'(s_1) >_{\text{kbo}} \sigma'(s_2) = t_1$.

2. If $s_1 >_{\text{kbo}} s_2$ holds for the second condition, then the root symbol f of s_1 and the root symbol g of s_2 are such that $f > g$. Obviously $\sigma(s_1)$ has root symbol f whereas $\sigma(s_2)$ as root symbol g, and thus $\sigma(s_1) >_{\text{kbo}} \sigma(s_2)$.

3. If $s_2 >_{\text{kbo}} s_2$ holds for the third condition then the root symbols for s_1, s_2, $\sigma(s_1)$ and $\sigma(s_2)$ are the same. Let $s_1 = f(s_1, \ldots, s_m)$ and $s_2 = f(t_1, \ldots, t_m)$. It holds that there exists $i \in [1, m]$ such that $s_1 = t_1$,...,$s_{i-1} = t_{i-1}$ and $s_i >_{\text{kbo}} t_i$. This implies $\sigma(s_1) = \sigma(t_1)$,...,$\sigma(s_{i-1}) = \sigma(t_{i-1})$ and by induction $\sigma(s_i) >_{\text{kbo}} \sigma(t_i)$ (since s_i is smaller than s_1). Thus $\sigma(s_1) > \sigma(s_2)$ holds.

(5) To show the subterm property, recall that $s >_{\text{kbo}} x$ for all variables x and terms $s \neq x$ that contain x. This, together with the fact that $>_{\text{kbo}}$ is closed under substitutions, obviously implies the subterm property.