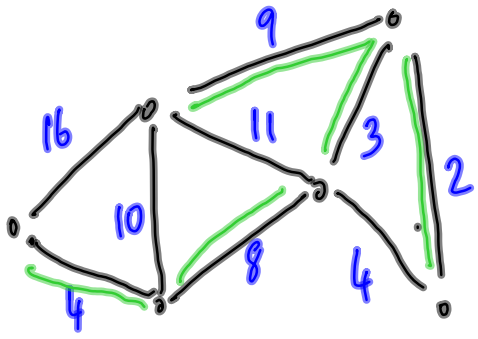
Minimum Cost Spanning Tree



Find a spanning tree of minimum cost

Greedy strategy [Kruskal's Algorithm]

Pick edges in ascending order of cost

Add ~~to tree~~ to forest if there is no loop

Facts:

A (spanning) tree on n vertices has $n-1$ edges

Any connected graph with n vertices & $n-1$ edges is a tree

Any pair (u,v) is connected by a unique path

Why is Kruskal's algorithm correct?

Greedy strategy

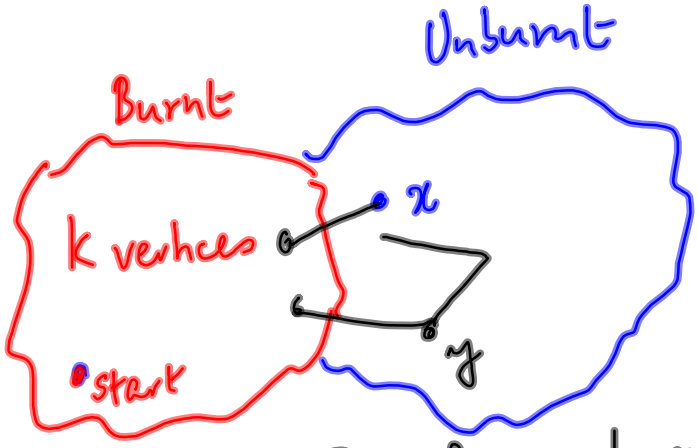
Make a locally optimal choice

Never backtrack

Why is Dijkstra's algorithm correct?

By induction

After k iterations: all burnt vertices are correctly labelled by their shortest paths



Assume x is burnt next. Path from s to x begins inside burnt portion & ends outside. Any other path leaves, goes to y then x, must have same length or more

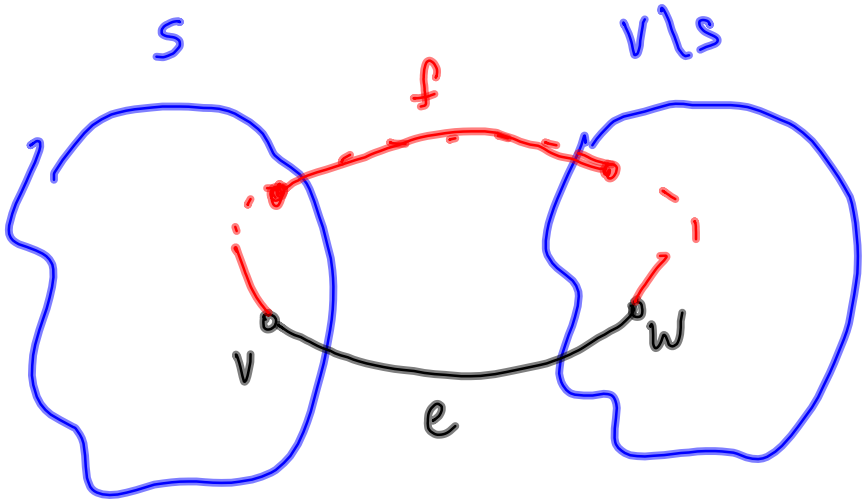
Difficult to argue Kruskal's correctness inductively
At intermediate stages, forest not tree

Give me another solution. I'll show you it is no better than mine. "Exchange argument"

CUT PROPERTY

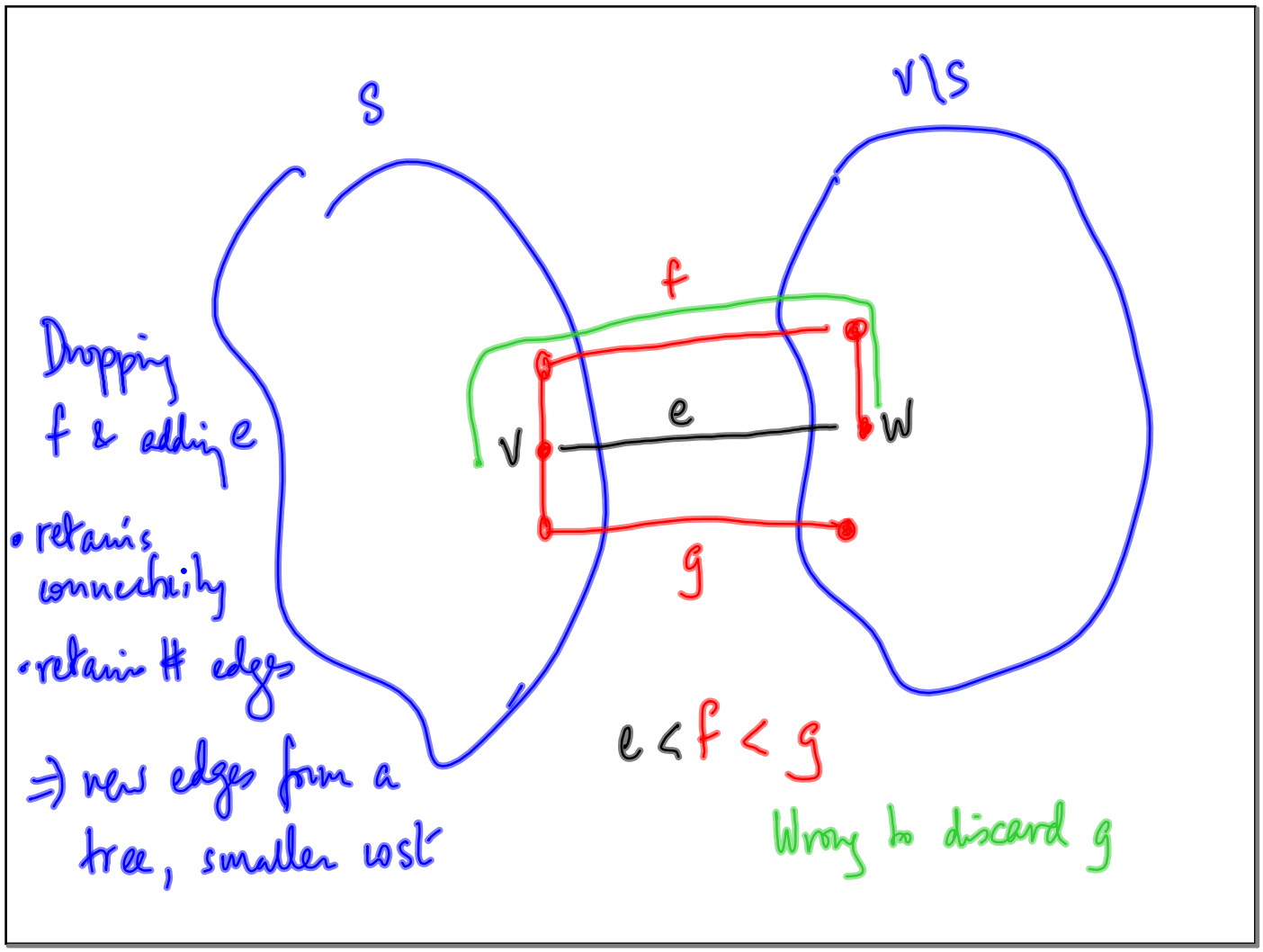
Assume all edges wts are distinct. [Can be weakened]
Consider any $S \subsetneq V$. Let e be the min cost edge connecting S to $V \setminus S$. e belongs to every MCST on G.

Proof:



Let $e=(v,w)$ be min cost edge, not in MCST

$\exists$ a tree connecting v,w. Path from v to w in tree goes from S to $V \setminus S$


S
V\S
f
e
v
w
g
Dropping f & adding e
• retains connectivity
• retains # edges
⇒ new edges form a tree, smaller cost
e < f < g
Wrong to discard g

Use CUT PROPERTY to show Kruskal's algo is correct

The algorithm examines edges in asc. order of cost

Another algorithm

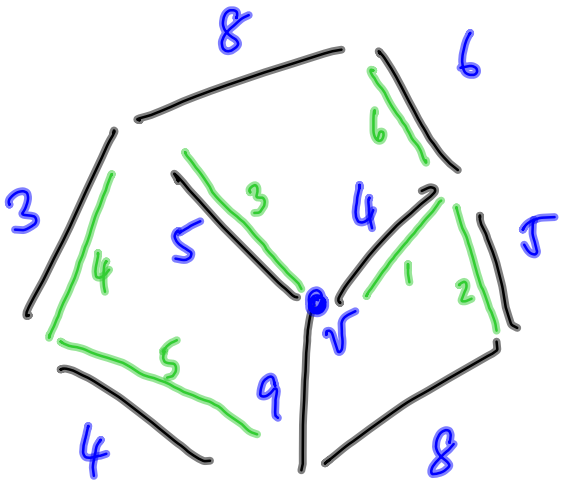
Start at any vertex v and incrementally build the smallest possible tree

$T = \{v\}$

while there are vertices not in T

pick $w \notin T$ s.t. $\exists v \in T$, (v,w) is smallest edge from T to $V \setminus T$

$T = T \cup \{w\}$ (i.e. add (v,w) to T)



Like Dijkstra, except criterion for adding ("burning") next vertex depends only on single edge weight, not cumulative distance

PRIM'S ALGORITHM

Like Dijkstra, use a heap to find min cost edge to pick up.

Implementing Kruskal's algorithm

- Sort edges initially $O(m \log m)$
- For each edge we consider, check if it forms a cycle

  Suffices to keep track of currently connected components

  An edge forms a cycle iff both end points lie in the same component

  Adding edge merges components

Given a set V, partitioned as $C_1, C_2 \ldots, C_k$

- FIND(u) - report the component of u
- UNION(u,v) - merge the component of u & v

UNION-FIND datastructure

Can $e=(v,w)$ be added to T? Check FIND(v)==FIND(w)

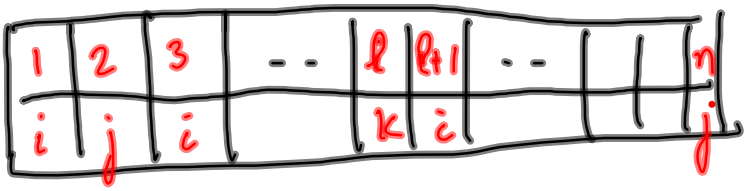
After adding e, UNION(v,w)

Initially each vertex is an isolated component

$V = \{1, 2, \ldots n\}$

Use a subset of $\{1, \ldots, n\}$ to name components

Initially. $Comp(i) = i \quad \forall i \in V$



Merge (3, ℓ) – renumber all i as k or. vice versa in second row

FIND(u) constant time

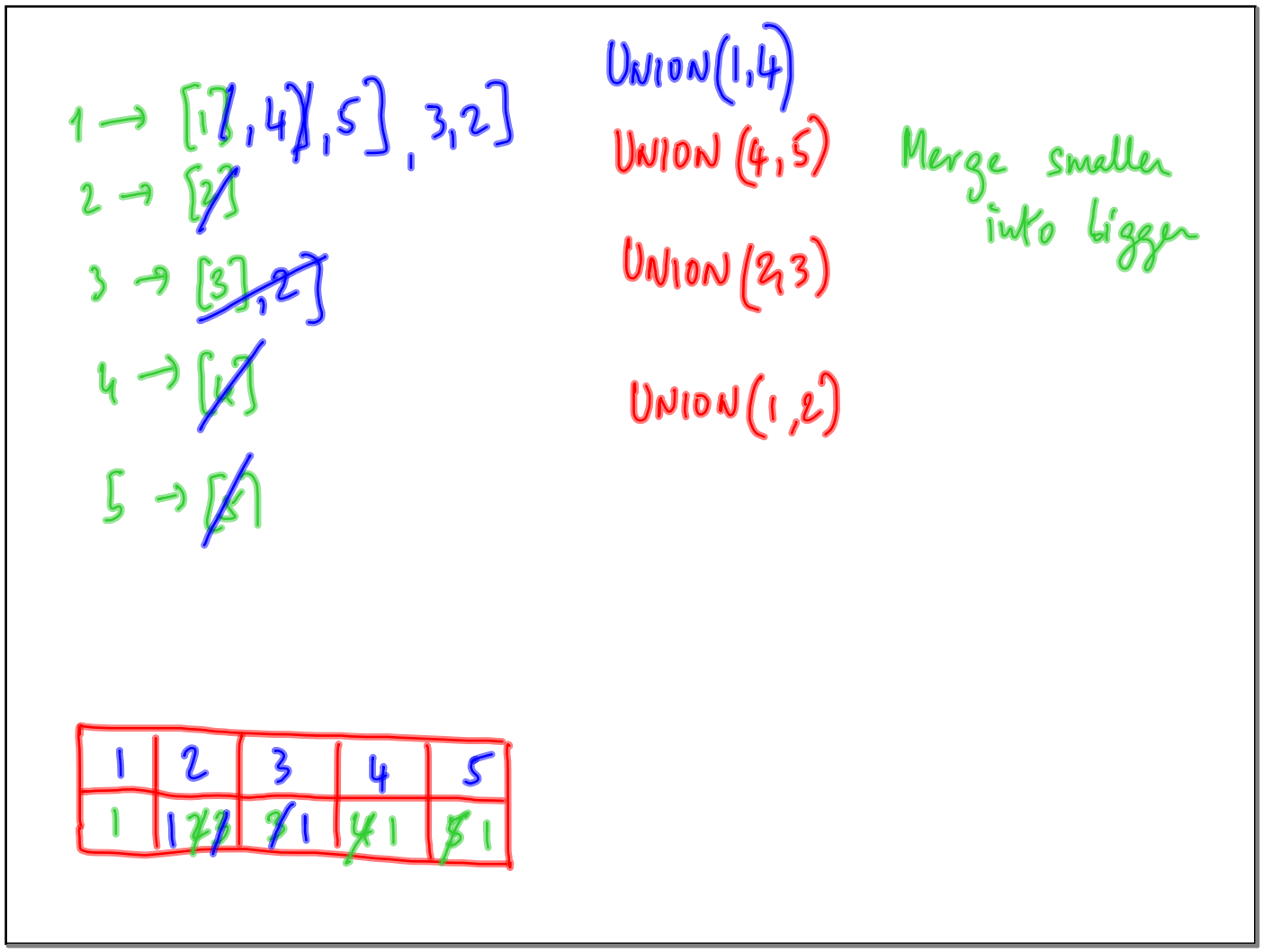
UNION(u,v) O(n) time

└ building MCST adds n-1 edges, so $O(n^2)$

Mild modification:

Keep each component as a list

Merge ⇒ combine lists


1 → [1,4,5], 3,2
2 → [2]
3 → [3],2
4 → [4]
5 → [5]
UNION(1,4)
UNION(4,5)
UNION(2,3)
UNION(1,2)
Merge smaller into bigger
1 2 3 4 5
1 1 1 1 1

$\forall v$ Component(v) doubles with each update

Each vertex is updated at most $\log n$ times

Overall, at most $O(n \log n)$ updates

AMORTISED ANALYSIS