Chennai Mathematical Institute
Probability Theory: January-April 2014

Review Problems

1. Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics of i.i.d. exponential random variables with parameter β.

   (a) Show that $X_{(r)}$ and $X_{(s)} - X_{(r)}$ are independent for any $s > r$.

   (b) Find the pdf of $X_{(r+1)} - X_{(r)}$.

2. A sequence of random variables $\{X_n\}$ is said to converge to X in **quadratic mean** or **mean-square** if
$$E(X_n - X)^2 \to 0$$
as $n \to \infty$.

   Let $X_1, \ldots X_n$ be iid $B(1,p)$ random variables. Show that $\bar{X}_n$ converges to p in quadratic mean.

3. Let $X_1, X_2, \ldots$ be i.i.d. $U(0,\theta)$. Show that $Y = X_{(n)}$ converges in probability to θ.

4. Suppose the conditional distribution of Y given $\Lambda = \lambda$ is Poisson with parameter λ. Assume that Λ has a $G(k,\beta)$ distribution where k is an integer. [There are two random variables here: Y which is discrete, and Λ which is continuous.]
$$\Lambda \sim G(k,\beta) \Rightarrow f_\Lambda(\lambda) = \frac{1}{\Gamma(k)\beta^k}\lambda^{k-1}e^{-\lambda/\beta}.$$
   Find the marginal pmf of Y.

5. Let $X_1, \ldots, X_2$ be iid with cdf
$$F(x) = x^\alpha \quad 0 < x < 1, \alpha > 0.$$
   Show that $\frac{X_{(1)}}{X_{(2)}}$ and $X_{(2)}$ are independent.

6. A discrete random variable X is said to have a power series distribution if its probability mass function is given by
$$P(X = x) = a(x)\theta^x / C(\theta), \qquad x = 0, 1, \cdots; \quad a(x) \geq 0; \quad \theta > 0$$
   Find the moment generating and probability generating functions of X.

7. Let X, Y have the joint pdf given by
$$f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty, 0 < y < \infty \\ 0 & o.w. \end{cases}$$

   (a) Find the marginal density of Y.

   (b) Find the conditional distribution of X given $Y = y$.

   (c) Find the conditional expectation $E(X|Y = y)$.

(d) Find $E(X)$ directly and by using (c).

8. Let X be a positive random variable with cdf given by

$$F(x) = (1 - exp^{-\lambda x})^{\alpha}; \qquad \alpha, \lambda, x > 0.$$

This is called the **exponentiated exponential** distribution.

(a) Show that this is a valid cdf.

(b) Find the probability density function of X.

(c) Find the mean and second moment of X assuming α is some positive integer k.

9. Let $X_1, \ldots, X_n$ be iid random variables with cdf given by

$$F(x) = \frac{1}{(1 + e^{-x})}, x \in \mathcal{R}$$

Find the distribution of $X_{(1)}$.

10. A random variable T is said to have a log-logistic distribution if $Y = ln(T)$ has the logistic distribution with pdf

$$\frac{1}{\sigma} \frac{exp[(y - \mu)/\sigma]}{\{1 + exp[(y - \mu)/\sigma]\}^2}, \qquad -\infty < y < \infty$$

Find the cdf and the hazard function for T. Describe the behaviour of the hazard function.

11. A public health survey is being planned in Chennai to estimate the true proportion of children, p, who have not been immunized. The proposed estimator, $\hat{p}$, is the proportion of children in the sample who have not been immunized.

The city government would like to estimate the true proportion to within a 5% error with 95% confidence. How large should the sample be?