Chennai Mathematical Institute
Probability Theory: January-April 2014
Mid-Semester Exam

Name ____

Answer all questions and show your work. Good Luck!

(1) A recent B. Com. graduate is planning to take the three Cost Accountancy Certificate exams this coming summer. She will take the first exam in June. If she passes the first exam, she will take the second exam in July, and if she passes that exam, she will take the third exam in September. If she fails an exam, she is not allowed to sit for any of the remaining exams.

The probability that she passes the first exam is 0.9. If she passes the first exam, the conditional probability that she passes the second exam is 0.8, and if she passes the first two exams, the conditional probability that she passes the third exam is 0.7.

(a) What is the probability that she passes all three exams.

(b) Given that she did not pass all three exams, what is the probability that she failed the second exam?

Please define the events and the required probabilities in terms of these events.

(2) A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period.

Calculate the probability that the participant was a heavy smoker.

(3) If $E_1, E_2, \ldots, E_n$ are independent events, show that

$$P[E_1 \cup E_2 \ldots \cup E_n] = 1 - \prod_{i=1}^{n}(1 - P(E_i))$$

(4) An insurance policy pays an individual Rs. 500 per day for up to 3 days of hospitalization and Rs. 250 per day for each day of hospitalization thereafter. The number of days of hospitalization, X, is a discrete random variable with probability mass function

$$p_X(k) = P(X = k) = \begin{cases} \frac{6-k}{15} & k = 1, 2, 3, 4, 5 \\ 0 & \textit{otherwise.} \end{cases}$$

Calculate the expected payment for hospitalization under this policy.

(5) Let X be a non-negative integer valued random variable. The **probability generating function** of X is defined as

$$P_X(t) = E(t^X) = \sum_k p_k t^k.$$

(a) Show that

$$\frac{1}{k!}\frac{d^k}{dt^k}P_X(t)\bigg|_{t=0} = p_k.$$

(b) Find the pgf for the Binomial and Poisson distributions.

(6) Consider $N = rk$ water samples randomly selected from various streams in the region. The water samples are tested for the presence of mercury and classified as either Positive (mercury detected) or Negative (no mercury detected). Let $X_i = 1$ if the $i-$th sample is Positive, 0 otherwise. Let θ represent the probability that a particular unit is positive and assume independence.

We randomly divide the N units into k groups each of size r, where r and k are both known. These are called **composite** samples. We assume that the composite sample can be tested simultaneously and will yield either a positive or negative result for mercury. If the test on the group is negative, all units in the group are negative for lead as well.

A test is performed on each of the k groups and the results recorded. Let Y denote the number of positive groups.

(a) Find the probability mass function of Y. What is this distribution?

(b) Why would researchers use composite sampling? Under what conditions do you think it would be effective?

(7) If A and B are independent events, show that A^c and B^c are also independent.

(8) The University of Timbuktu advertises positions in the Department of Physics and in the Department of Geography, and only those departments. Five men apply for the positions in Physics and one is hired, and eight women apply and two are hired. In the Geography Department eight men apply and six are hired, and five women apply and four are hired. Compute the success rate for male and female candidates by department and the overall success rate across the departments. Can you explain the discrepancy?

(9) (Bonus Question) A random variable X is said to have a log-series distribution if its pmf is given by

$$p_X(k) = c\ \frac{p^k}{k}, \qquad k = 1, 2, \ldots$$

(a) Find c.

(b) Find $E(X)$.

Hint: Consider the expansion

$$(1-x)^{-1} = 1 + x + x^2 + \dots.$$

Integrate both sides.