Programming Language Concepts: Lecture 18

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One step reduction

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One step reduction

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- New reduction rule η

$$\lambda x.(Mx) \to_\eta M$$

- Given basic rules β, η, ..., we are allowed to use them "in any context"
- Define a one step reduction relation $\to$ inductively

$$\frac{M \to_x M'}{M \to M} \quad x \in \{\beta, \eta, \ldots\} \qquad \frac{M \to M'}{\lambda x.M \to \lambda x.M'} \qquad \frac{M \to M'}{MN \to M'N} \qquad \frac{N \to N'}{MN \to MN'}$$

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- If a term has a normal form, can we always find it?

Normal forms . . .

Does every term reduce to a normal form?

- Consider $(\lambda x.xx)(\lambda x.xx)$

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- Consider $\langle \mathit{False} \rangle \Omega = (\lambda yz.z)((\lambda x.xx)(\lambda x.xx))$

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- Choice of reduction strategies may determine whether a normal form is reached . . .
- . . . but the question is, can more than one normal form be reached?

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- We have seen how to encode recursive functions in λ-calculus
- Given a recursive function f and an argument n, we cannot determine, in general, if computation of $f(n)$ terminates
- Computing $f(n)$ is equivalent to asking if $\langle f \rangle \langle n \rangle$ achieves a normal form

Normal forms . . .

Can a term reduce to more than one normal form, depending on order reduction strategy?

- Define an equivalence relation $\leftrightarrow$ on λ-terms

$$M \leftrightarrow N \text{ iff } \exists P.\ P \rightarrow^* M, P \rightarrow^* N$$

$M \leftrightarrow N$ if both M and N can be obtained by reduction from a common "ancestor" P

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- $\leftrightarrow$ is the symmetric transitive closure of $\rightarrow^*$

$$\frac{M \rightarrow^* N}{M \leftrightarrow N} \quad \frac{M \leftrightarrow N}{N \leftrightarrow M} \quad \frac{M \leftrightarrow N, N \leftrightarrow P}{M \leftrightarrow P}$$

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$$\frac{M \rightarrow^* N}{M \leftrightarrow N} \quad \frac{M \leftrightarrow N}{N \leftrightarrow M} \quad \frac{M \leftrightarrow N, N \leftrightarrow P}{M \leftrightarrow P}$$

- In general, for any reflexive, transitive relation R, can define the symmetric, transitive closure $\overset{R}{\leftrightarrow}$

Church-Rosser Theorem

Diamond property or Church-Rosser property

- Let R be any reflexive, transitive relation (such as $\rightarrow^*$)
- R has the diamond property if, whenever $X \; R \; Y$ and $X \; R \; Z$ there is W such that $Y \; R \; W$ and $Z \; R \; W$

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Theorem [Church-Rosser]

Let R be Church-Rosser. Then $M \overset{R}{\leftrightarrow} N$ implies there exists Z, $M \; R \; Z$ and $N \; R \; Z$

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Proof By induction on the definition of $\overset{R}{\leftrightarrow}$

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Let R be a reduction relation that is Church-Rosser. Then a term can have at most one normal form with respect to R

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Proof By picture

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Two possible reductions

- $(\lambda x.xx)((\lambda x.x)(\lambda x.x)) \rightarrow$
  $((\lambda x.x)(\lambda x.x))((\lambda x.x)(\lambda x.x))$ (Outermost)
- $(\lambda x.xx)((\lambda x.x)(\lambda x.x)) \rightarrow ((\lambda x.xx)(\lambda x.x)$ (Innermost)

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From second option, in one step we get

$$(\lambda x.xx)(\lambda x.x) \rightarrow ((\lambda x.x)(\lambda x.x))$$

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From second option, in one step we get

$$(\lambda x.xx)(\lambda x.x) \rightarrow ((\lambda x.x)(\lambda x.x))$$

Can reach this term from the first option as well, but it requires two steps!

Church-Rosser Theorem

Solution: Define a new notion of one step reduction $\twoheadrightarrow$ such that

- This new reduction is Church-Rosser.
- Its reflexive, transitive closure is equal to $\longrightarrow^*$.

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Define $\twoheadrightarrow$ as follows.

$$M \twoheadrightarrow M \qquad \frac{M \twoheadrightarrow M'}{\lambda x.M \twoheadrightarrow \lambda x.M'}$$

$$\frac{M \twoheadrightarrow M', N \twoheadrightarrow N'}{MN \twoheadrightarrow M'N'} \qquad \frac{M \twoheadrightarrow M', N \twoheadrightarrow N'}{(\lambda x.M)N \twoheadrightarrow M'\{x \leftarrow N'\}}$$

- $\twoheadrightarrow$ combines nonoverlapping $\longrightarrow$ reductions into one parallel step

Recursive definitions

Suppose $F = \lambda x_1 x_2 \ldots x_n E$, where where E contains an occurrence of F

- Choose a new variable f
- Convert E to E^* replacing every F in E by ff
  - If E is of the form $\cdots F \cdots F \cdots$ then E^* is $\cdots (ff) \cdots (ff) \cdots$.

Now write

$$\begin{aligned} G &= \lambda f x_1 x_2 \ldots x_n . E^* \\ &= \lambda f x_1 x_2 \ldots x_n . \cdots (ff) \cdots (ff) \cdots \end{aligned}$$

Then

$$GG = \lambda x_1 x_2 \ldots x_n . \cdots (GG) \cdots (GG) \cdots$$

- GG satisfies the equation defining F
- Write $F = GG$, where $G = \lambda f x_1 x_2 \ldots x_n . E^*$.

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- Due to Turing — Θ

Terms without normal forms

Are all terms without normal forms equally "meaningless"?

Can we define an equivalence $\approx$ on λ-terms such that:

- $(\lambda x M)N \approx M\{x \leftarrow N\}$—that is, $\approx$ the equivalence induced by the β reduction.

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- $(\lambda x M)N \approx M\{x \leftarrow N\}$—that is, $\approx$ the equivalence induced by the β reduction.
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- Functions that are equated by $\approx$ yield equivalent results for the same arguments. That is, if $M \approx N$ then for all R, $MR \approx NR$.

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Consider $FX\langle true\rangle$ and $FX\langle false\rangle$

- $FX\langle true\rangle \to \text{if } \langle T\rangle \text{ then } X \text{ else } (FX\langle true\rangle) \to X$.

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Consider $FX\langle true\rangle$ and $FX\langle false\rangle$

- $FX\langle true\rangle \to \text{if } \langle T\rangle \text{ then } X \text{ else } (FX\langle true\rangle) \to X$.
- $FX\langle false\rangle \to \text{if } \langle F\rangle \text{ then } X \text{ else } (FX\langle false\rangle) \to FX\langle false\rangle$.

Terms without normal forms

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\begin{aligned}
FZ &\rightarrow (\lambda xb.(\text{if } b \text{ then } x \text{ else } (Fxb)))Z \\
&\rightarrow \lambda b.(\text{if } b \text{ then } Z \text{ else } (FZb)) \\
&\rightarrow \lambda b.(\text{if } b \text{ then } Z \text{ else } G), \text{where } G = \text{if } b \text{ then } Z \text{ else } (fZB) \\
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- $FX \approx FY$ implies $FXM \approx FYM$ for all M

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- FZ does not terminate for any $Z \Rightarrow FX \approx FY$ for all X, Y
- $FX \approx FY$ implies $FXM \approx FYM$ for all M
- $FX\langle true\rangle \approx FY\langle true\rangle$
- $FZ\langle true\rangle \to Z$ for all Z, so $X \approx Y$ for all X and Y!