Programming Language Concepts: Lecture 15

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  - Alonzo Church

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  - All sorting functions are the same!

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  - Extensional — graph completely defines the function
- An extensional definition is not suitable for computation
  - All sorting functions are the same!
- Need an intensional definition
  - How are outputs computed from inputs?

λ-calculus: syntax

- Assume a set Var of variables
- Set Λ of lambda expressions is given by

$$\Lambda = x \mid \lambda x.M \mid MM'$$

where $x \in Var$, $M, M' \in \Lambda$.

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- MM' : Application
  - Apply the function M to the argument M'

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- In an untyped world, some data is meaningful
- Functions manipulate meaningful data to yield meaningful data
- Can also apply functions to non-meaningful data, but the result has no significance

The computation rule β

- Basic rule for computing (rewriting) is called β

$$(\lambda x.M)M' \rightarrow_\beta M\{x \leftarrow M'\}$$

  - $M\{x \leftarrow M'\}$: substitute free occurrences of x in M by M'

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- β is the only rule we need!
- MM' is meaningful only if M is of the form $\lambda x.M''$
  - Cannot do anything with expressions like xx

Variable capture

- Consider $(\lambda x.(\lambda y.xy))y$

Variable capture

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- β yields $\lambda y.yy$
  - The y substituted for inner x has been "confused" with the y bound by λy
- Rename bound variables to avoid capture

$$(\lambda x.(\lambda y.xy))y = (\lambda x.(\lambda z.xz))y \rightarrow_\beta \lambda z.yz$$

- Renaming bound variables does not change the function
  - $f(x) = 2x + 5$ vs $f(z) = 2z + 5$

Variable capture

Formally, bound and free variables are defined as

- $FV(x) = \{x\}$, for any variable x
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When we apply β to MM', assume that we always rename the bound variables in M to avoid "capturing" free variables from M'.

Encoding arithmetic

In set theory, use nesting depth to encode numbers

- Encoding of n: $\langle n \rangle$
- $\langle n \rangle = \{\langle 0 \rangle, \langle 1 \rangle, \ldots, \langle n-1 \rangle\}$

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In λ-calculus, encode n by number of times we apply a function

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Church numerals

$$\begin{aligned} \langle 0 \rangle &= \lambda fx.x \\ \langle n+1 \rangle &= \lambda fx.f(\langle n \rangle fx) \end{aligned}$$

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Note that $\langle 0 \rangle gy \rightarrow_\beta (\lambda x.x)y \rightarrow_\beta y$.
Hence

$$\langle 1 \rangle = \ldots = \lambda fx.(f(\underbrace{(\lambda fx.x)fx}_{\text{apply } \beta})) \rightarrow_\beta \lambda fx.(fx)$$

So $\langle 1 \rangle gy \rightarrow_\beta (\lambda x.(gx))y \rightarrow_\beta gy$

Church numerals . . .

$$\langle 2\rangle = \lambda fx.f(\langle 1\rangle fx) = \lambda fx.(f(\underbrace{\lambda fx.(fx)fx}_{\text{apply } \beta})) \rightarrow_\beta \lambda fx.(f(fx))$$

so,

$$\langle 2\rangle gy \rightarrow_\beta \lambda x.(g(gx))y = g(gy)$$

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- Let $g^k y$ denote $g(g(\ldots(gy)))$ with k applications of g to y
- Show by induction that

$$\langle n\rangle = \lambda fx.f(\langle n{-}1\rangle fx) \rightarrow_\beta \ldots \rightarrow_\beta \lambda fx.(f^n x)$$

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