Programming Language Concepts: Lecture 17

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Recursive functions

Recursive functions [Gödel]

Initial functions

- Zero: $Z(n) = 0$.
- Successor: $S(n) = n+1$.
- Projection: $\Pi_i^k(n_1, n_2, \ldots, n_k) = n_i$

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Composition Given $f : \mathbb{N}^k \to \mathbb{N}$ and
$g_1, g_2, \ldots, g_k : \mathbb{N}^h \to \mathbb{N}$,

$$f \circ (g_1, g_2, \ldots, g_k)(n_1, n_2, \ldots, n_h) =$$
$$f(g_1(n_1, n_2, \ldots, n_h), g_2(n_1, n_2, \ldots, n_h), \ldots, g_k(n_1, n_2, \ldots, n_h))$$

Recursive functions . . .

Primitive recursion Given $g : \mathbb{N}^k \to \mathbb{N}$ and
$h : \mathbb{N}^{k+2} \to \mathbb{N}$

define $f : \mathbb{N}^{k+1} \to \mathbb{N}$ by primitive recursion as follows:

$$\begin{aligned} f(0, n_1, n_2, \ldots, n_k) &= g(n_1, n_2, \ldots, n_k) \\ f(n{+}1, n_1, \ldots, n_k) &= h(n, f(n, n_1, n_2, \ldots, n_k), n_1, \ldots, n_k) \end{aligned}$$

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Minimalization

Given $g : \mathbb{N}^{k+1} \to \mathbb{N}$, define $f : \mathbb{N}^k \to \mathbb{N}$ by minimalization from g

$$f(n_1, n_2, \ldots, n_k) = \mu n.(g(n, n_1, n_2 \ldots, n_k) = 0)$$

where $\mu n.P(n)$ returns the least natural number n such that $P(n)$ holds

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- Clearly, $f(n) = snd(t(n))$

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- . . . and $f(n) = snd(t(n)) = snd(step^n(0, g))$
- Will require constructions for building pairs and decomposing them using *fst* and *snd*

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An encoding for pairs

- $\langle pair \rangle \equiv \lambda xyz.(zxy)$.
- $\langle fst \rangle \equiv \lambda p.(p(\lambda xy.x))$.
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Use primitive recursive defn of $t(n)$ to encode $step$

$$\langle step \rangle \equiv \lambda x. \langle pair \rangle \ (\langle succ \rangle (\langle fst \rangle x)) \ (\langle h \rangle \ (\langle fst \rangle x) \ (\langle snd \rangle x)).$$

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Check that $\langle step \rangle\ (\langle pair \rangle \langle n \rangle \langle f(n) \rangle) \rightarrow^* \langle pair \rangle\ \langle n{+}1 \rangle \langle f(n{+}1) \rangle$

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Hence

$$\langle t \rangle \equiv \lambda y.y\ \langle step \rangle (\langle pair \rangle \langle 0 \rangle \langle g \rangle)$$

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$$\langle t \rangle \equiv \lambda y.y\ \langle step \rangle (\langle pair \rangle \langle 0 \rangle \langle g \rangle)$$

Check that for all n, $\langle t \rangle \langle n \rangle \rightarrow^* pair\ \langle n \rangle\ \langle f(n) \rangle$.

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Back to primitive recursion . . .

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Back to primitive recursion . . .

Clearly $f(n) = snd(t(n))$

$$\langle f \rangle \equiv \lambda y.\langle snd \rangle(\langle t \rangle y)$$

Collapsing all the steps we have seen above, we have an expression $\langle PR \rangle$ that constructs a primitive recursive definition from the functions g and h:

$$\langle PR \rangle \equiv \lambda hgy.\langle snd \rangle(y\ (\lambda x.\langle pair \rangle\ (\langle succ \rangle\ (\langle fst \rangle\ x))\ (h(\langle fst \rangle\ x))\ (\langle snd \rangle\ x)))(\langle pair \rangle\ \langle 0 \rangle\ g)$$

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Recursive functions . . .

Minimalization

- To evaluate

$$f(n_1, n_2, \ldots, n_k) = \mu n.(g(n, n_1, n_2 \ldots, n_k) = 0)$$

we go back to the idea of computing a while loop

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- GG satisfies the equation defining F
- Write $F = GG$, where $G = \lambda f x_1 x_2 \ldots x_n . E^*$.

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Encoding Booleans

- $\langle True \rangle \equiv \lambda xy.x$
- $\langle False \rangle \equiv \lambda xy.y$
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- Want to define $f : \mathbb{N}^k \to \mathbb{N}$ by minimalization from a $g : \mathbb{N}^{k+1} \to \mathbb{N}$
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Still need to define $\langle \mathit{iszero} \rangle$

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By induction, for $n > 0$ . . .

$$\langle iszero \rangle\ \langle n \rangle \to_\beta^* (\lambda z.\langle false \rangle)^n\ \langle true \rangle \to_\beta^* \langle false \rangle$$