Advanced Complexity

TD n°3

Aliaume Lopez

October 9, 2019

Exercise 1 : Restrictions of the SAT problem

1. Let 3-SAT be the restriction of SAT to clauses consisting of at most three literals (called 3-clauses). In other words, the input is a finite set S of 3-clauses, and the question is whether S is satisfiable. Show that 3-SAT is NP-complete for logspace reductions (assuming SAT is).
2. Let 2-SAT be the restriction of SAT to clauses consisting of at most two literals (called 2-clauses). Show that 2-SAT is in P, using proofs by resolution.
3. Show that 2-UNSAT (i.e, the unsatisfiability of a set of 2-clauses) is NL-complete.
4. Conclude that 2-SAT is NL-complete.

Exercise 2 : Space hierarchy theorem

Using a diagonal argument, prove that for two space-constructible functions f and g such that $f(n) = o(g(n))$ (and as always $f, g \geq log$) we have $\mathsf{SPACE}(f(n)) \subsetneq \mathsf{SPACE}(g(n))$.

Exercise 3 : My very first PSPACE-complete problem

Show that the following problem is PSPACE-complete (not assuming anything about QBF) :

— INPUT : a Turing Machine M and a word w and a number t written in unary

— QUESTION : does M accepts w within space t ?

Exercise 4 : Polylogarithmic space

Let $\mathsf{polyL} = \cup_{k\in\mathbb{N}} \mathsf{SPACE}(log^k(n))$. Show that $\mathsf{polyL} \neq \mathsf{P}$.

1. Show that polyL does not have a complete problem ?
2. Does PSPACE, constructed in the same fashion have one ?
3. Does P have a complete problem ?
4. Deduce $\mathsf{polyL} \neq P$.

Exercise 5 : Padding argument

1. Show that if $\mathsf{DSPACE}(n^c) \subseteq \mathsf{NP}$ for some $c > 0$, then $\mathsf{PSPACE} \subseteq \mathsf{NP}$.
2. Deduce that $\mathsf{DSPACE}(n^c) \neq \mathsf{NP}$.

Exercise 6 : Closure under morphisms

Given a finite alphabet Σ, a function $f : \Sigma^* \to \Sigma^*$ is a morphism if $f(\Sigma) \subseteq \Sigma$ and for all $a = a_1 \cdots a_n \in \Sigma^*$, $f(a) = f(a_1) \cdots f(a_n)$ (f is uniquely determined by the value it takes on Σ).

1. Show that NP is closed under morphisms, that is : for any language $L \in \mathsf{NP}$, and any morphism f on the alphabet of L, $f(L) \in \mathsf{NP}$.
2. Show that if P is closed under morphisms, then $\mathsf{P} = \mathsf{NP}$.

Exercise 7 : Unary Languages

Collapsing P and NP...

1. Prove that if a unary language is NP-complete, then $\mathsf{P} = \mathsf{NP}$.
   Hint : consider a reduction from SAT to this unary language and exhibit a polynomial time recursive algorithm for SAT
2. Prove that if every unary language in NP is actually in P, then $\mathsf{EXP} = \mathsf{NEXP}$.

Exercise 8 : A translation result

Show that if $\mathsf{P} = \mathsf{PSPACE}$, then $\mathsf{EXPTIME} = \mathsf{EXPSPACE}$.

Exercise 9 : A strict hierarchy between P and NP

Show that if $\mathsf{P} \neq \mathsf{NP}$ then there exists infinitely many classes C_i such that

$$\mathsf{P} \subsetneq C_1 \subsetneq \cdots \subsetneq C_k \subsetneq \cdots \subsetneq \mathsf{NP} \tag{1}$$