1 Monoids and MSO

Exercise 1 (Monoids and Regularity). We say that a language is recognized by a finite monoid $M$ if there exists a morphism $\mu : A^* \to M$ and a subset $P \subseteq M$ such that $L = \mu^{-1}(P)$.

1. Prove that a language is regular if and only if it is recognized by a finite monoid.

2. Use the above result to conclude that regular languages are closed under the following operations:
   - union,
   - intersection,
   - complement,
   - reversal,
   - concatenation

3. Prove that the class of regular languages is closed under radicals $\sqrt{L} := \{w \in A^* \mid ww \in L\}$.

4. Prove that the class of regular language is closed under Kleene star.

Exercise 2 (From MSO to Monoids). Let $\varphi$ be an MSO sentence over finite words. Prove that there exists a monoid $M$ and a function $f : A^* \to M$ and a subset $P \subseteq M$ such that $w \models \varphi$ if and only if $f(w) \in P$.

Can you adapt the construction in the case of MSO formulas?

$\triangleright$ Hint 1

Exercise 3 (From Monoids to MSO). Let $M$ be a finite monoid and $m \in M$. Construct an MSO formula $\varphi_m(x, y)$ over $M^*$ that accepts all pairs $x < y$ such that the factor $w[x : y]$ evaluates to $m$.

If the monoid is aperiodic, can you write this formula in FO?
2 Factorisation Forests

Exercise 4 (Baby Factorisation Forest). Let $M$ be a finite monoid. Prove that there exists a constant $N$ such that for every $w \in M^*$ with $|w| \geq N$, there exist $v_0, v_1 \in M^*$, and $u \in M^+$ such that $w = v_0uv_1$ and $u$ is an idempotent element of $M$.

Exercise 5 (First-order Factorisation Forests). Let $M$ be a finite aperiodic monoid. Prove that there exists a constant $N$ such that for every $w \in M^*$, one can build a factorisation of $w$ of depth at most $N$, where idempotent products are replaced by constant products.

Exercise 6 (Pumping lemma for regular functions). Let $f$ be a regular function. Prove that there exists $N \geq 0$ such that for all $w \in A^*$ with $|w| \geq N$, there exist $v_0, v_1 \in A^*, u \in A^+, n \geq 0, \alpha_0, \ldots, \alpha_n \in B^*, \beta_1, \ldots, \beta_n \in B^+$ such that $w = v_0uv_1$ and

$$f(v_0u^{X+1}v_1) = \alpha_0\beta_1^X\alpha_1 \ldots \beta_n^X\alpha_n, \quad \text{for all } X \geq 0.$$  

▷ Solution 1 (Solution)

Exercise 7 (Efficient Query Evaluation). Let $q \in \mathbb{N}$. Provide a linear-time computation of a data-structure over a word $w$ allowing for constant-time answer to $\text{MSO}$ queries of quantifier depth at most $q$.

▷ Hint 2
A Hints

Hint 1 (Exercise 2 Use automata theory). At least for the first part, you can use the fact that $\varphi$ defines a regular language.

Hint 2 (Exercise 7 Use factorisation forests). Construct a factorisation forest of the monoid of $\mathcal{MSO}^q$ types.
B Solutions

Solution 1 [Solution to Exercise 6]. Without loss of generality, we assume that \( Q = Q^- \cup Q^+ \), where states in \( Q^- \) are always doing left transitions, while states in \( Q^+ \) are always doing right transitions. We define the transition monoid of \( f \) as follows: \( M := Q^\rightarrow \). The intended semantics is that given a state \( q \in Q \) and a word \( u \), the transition performed by \( u \) is given by the first state reached by \( f \) outside of the word \( u \).