$\mathbb{Z}$-POLYREGULAR FUNCTIONS

DECIDING APERIODICITY OF (POLY-)REGULAR FUNCTIONS

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Post-doctoral student at the automata team of MIMUW, Warsaw, under the supervision of Mikołaj Bojańczyk.

I am no expert on transducers!

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- Finite Model Theory (first order logic)

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- Finite Model Theory (first order logic)
- Well-quasi-orderings (combinatorics)
- Noetherian spaces (topology)

Aperiodicity, Star-free, and First-order logic

Enter a regular language

Finite Automaton

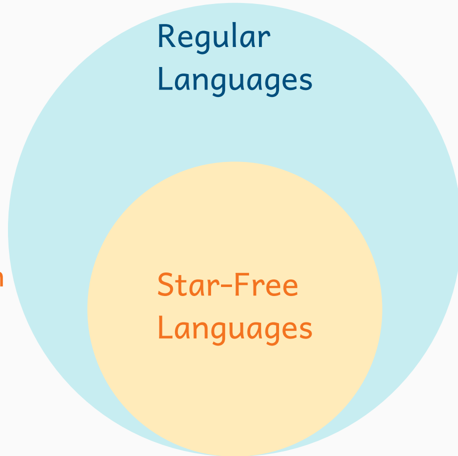
Finite Monoid

MSO Sentence

Counter-free Automaton

Aperiodic Monoid

FO Sentence



Decidability of the membership problem follows from the effective equivalence with aperiodic monoids [Sch65].

Regularity and aperiodicity for regular languages

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Finite Monoid

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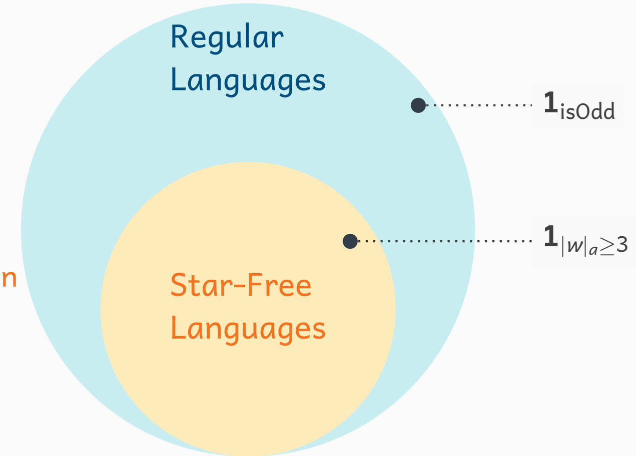

Counter-free Automaton

Aperiodic Monoid

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Aperiodicity, Star-free, and First-order logic

What about functions?

Brief overview of aperiodicity for functions (or relations)

$$L \colon \Sigma^\star \to \mathbb{B}$$

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Computational Model	Decidable aperiodicity
$f \subseteq (\Sigma \times \Gamma)^\star$ is a regular language	✓[Sch65]
f is sequential	✓[Cho03]
f is rational	✓[FGL16]
f is regular	$\approx$ [Boj14]
f is polyregular	?

In this talk:

polyregular functions

Aperiodicity, Star-free, and First-order logic

Simplifying until it trivialises

Arbitrary polyregular functions

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Unary output polyregular functions $\Gamma = \{1\}$

$$f\colon \Sigma^\star \to \{1\}^\star \simeq (\mathbb{N}, +)$$

Also known as $\mathbb{N}$-polyregular functions.

Careful choice of output

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$\mathbb{Z}$-output polyregular functions

$$f\colon \Sigma^\star \to \{+1, -1\}^\star$$

Casted to $(\mathbb{Z}, +)$ by post-composition with $\sum$.

Simplifications

The many advantages of $\mathbb{Z}$-output

- Commutative ouptut! (no ordering needed)
- Invertible output! (bounded backtracking is possible)
- Simpler definitions! (to be seen)
- Reduces to *counting* (rational series)

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Disatvantages

- The function $\sum : \{-1, +1\}^\star \to \mathbb{Z}$ is *not* regular.
- Non trivial compensations arise in the output.

$\mathbb{Z}$-polyregular functions

From a database person's perspective

From languages to functions via free variables

Theorem (Languages and MSO [Büc60])

A language L is regular iff there exists a ***sentence*** $\varphi \in \mathrm{MSO}$ *such that* $L = \mathbf{1}_{\varphi}$.

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What if φ was not a *sentence*?

Definition (Counting first order valuations)

$$\#\left[\varphi(\vec{x})\right] : w \mapsto \#\left[\{\vec{a} \in w \mid w, \vec{a} \models \varphi(\vec{x})\}\right] \quad .$$

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Remark

This is connected to “counting automata” [Sch62].

$$\mathbb{Z}\mathsf{P} := \mathsf{Lin}_{\mathbb{Z}}\left(\{\#\,[\varphi(\vec{x})] \mid \varphi(\vec{x}) \in \mathsf{MSO}\}\right)$$

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$\mathbb{Z}$-polyregular functions

Pop Quizz

Which of these functions are $\mathbb{Z}$-polyregular?

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- $w \mapsto (-1)^{|w|} \times |w|$

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Functions $f \in \mathbb{Z}\mathrm{P}$ have polynomial growth rate

For all $f \in \mathbb{Z}\mathrm{P}_k$,

$$|f(w)| = \mathcal{O}(|w|^k)$$

A frequently redefined concept?

Name	Reference
Finite Counting Automata	[Sch62]
Rational series of polynomial growth	[BR11]
Rational series without kleene star	—
Weighted automata of polynomial ambiguity	[KR13; CDTL23]
Polyregular functions (post composed with $\sum$)	[BKL19]

Membership is decidable and conversions are effective between these classes [see, e.g. CDTL23].

Aperiodicity

Which is what we cared about?

Pop Quizz (again?!)

Which of the following functions **should be** aperiodic?

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- $w \mapsto (-1)^{|w|} \times |w|$

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- $w \mapsto (-1)^{|w|} \times |w|$ ✗
- $w \mapsto (|w|_a - |w|_b)^2$ ✓

Please notice

For the last function, the pre-image of $\{0\}$ is not a regular language ...

Following the definition of Droste *and* Gastin [DG19]

The function $w \mapsto (-1)^{|w|}$ is *aperiodic*.

Following the definition of Droste and Gastin [DG19]

The function $w \mapsto (-1)^{|w|}$ is aperiodic.

Following the definition of Reutenauer [Reu80]

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Existing notions of aperiodicity

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Following the definition of Reutenauer [Reu80]

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I tricked you to *agree* with me.

Aperiodicity

A reasonable notion of aperiodicity?

Star-free $\mathbb{Z}$-polyregular functions

$$\mathbb{Z}\mathsf{SF} := \mathsf{Lin}_{\mathbb{Z}}\left(\{\#[\varphi(\vec{x})] \mid \varphi(\vec{x}) \in \mathsf{FO}\}\right)$$

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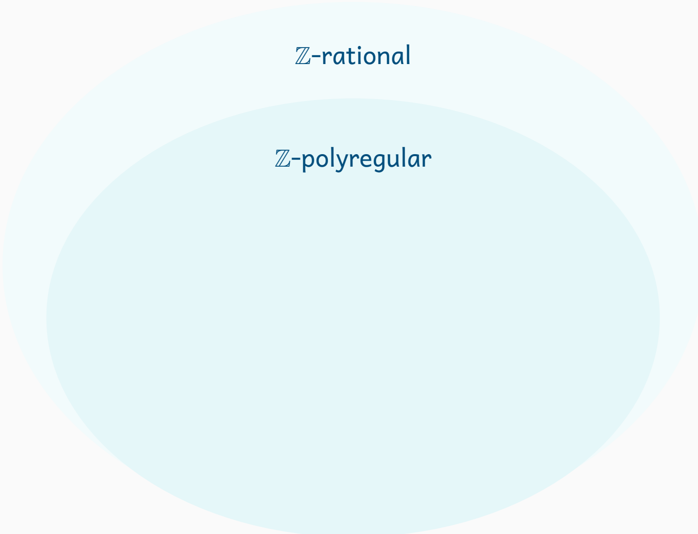
Our Results: Effective decision procedures.



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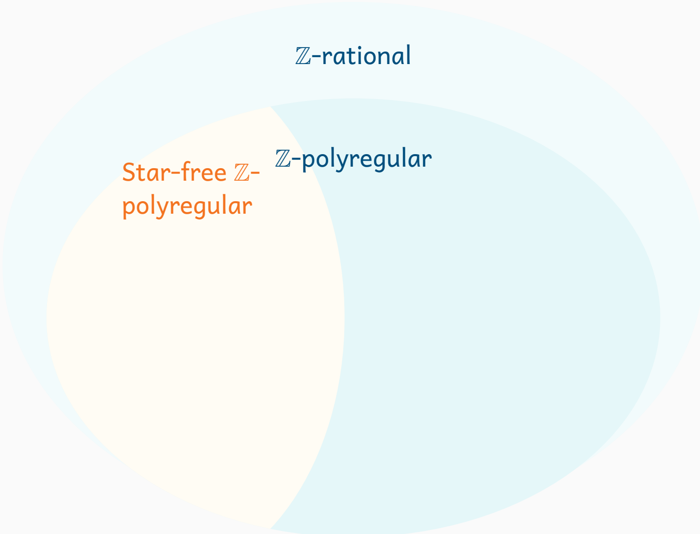

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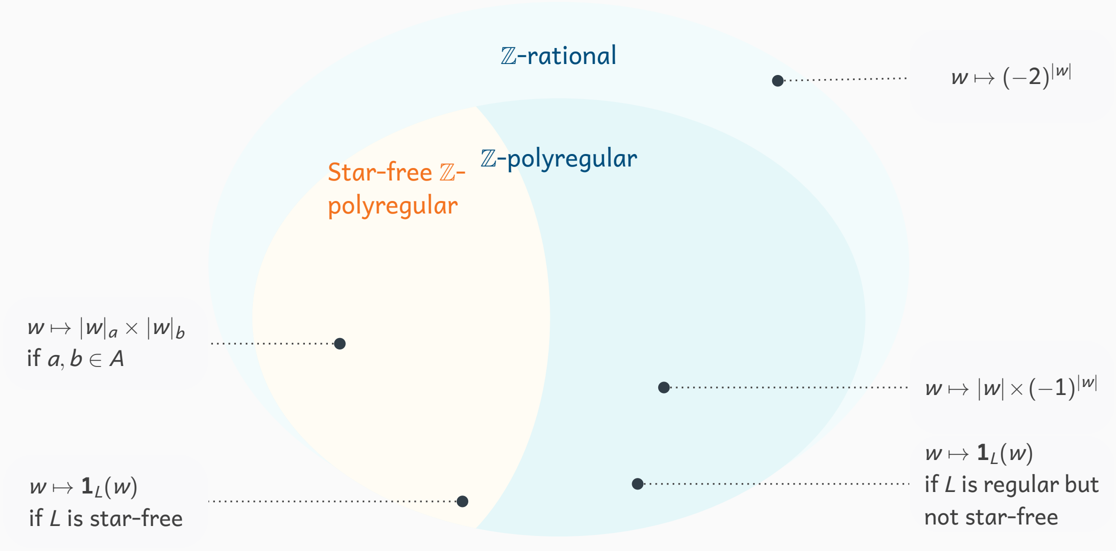

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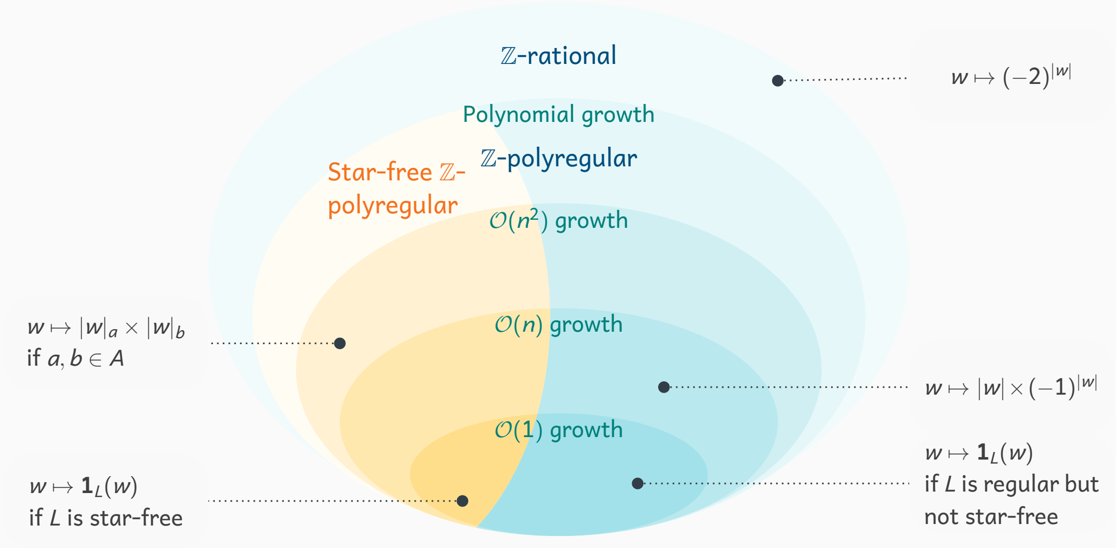

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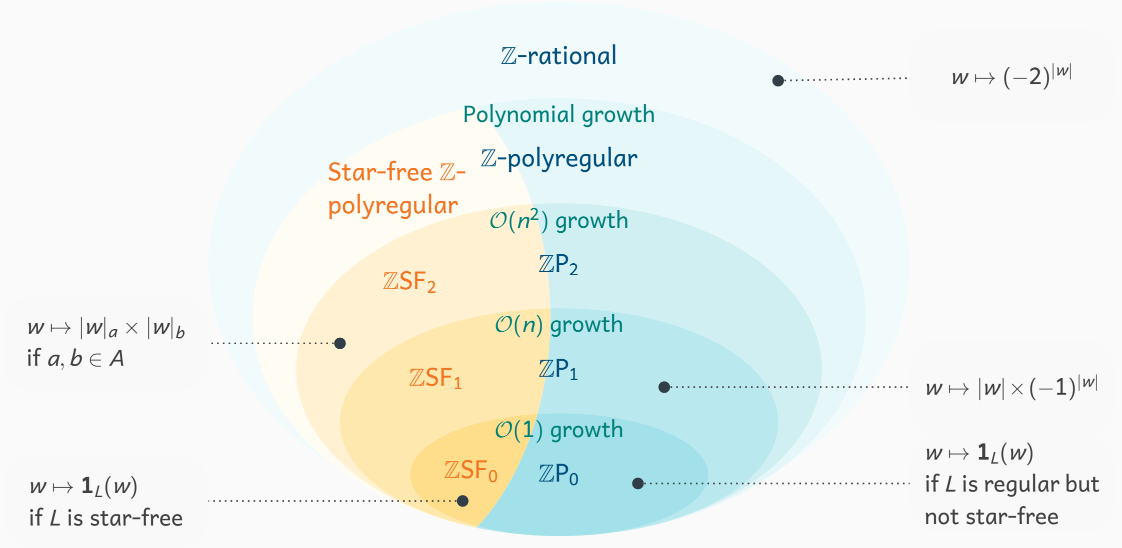

Proofs?

Deciding growth rate

It is non trivial

$$f(w) := \#\,[\mathrm{isOdd}(x)] - \#\,[\mathrm{isEven}(x)] \quad \in \mathbb{Z}\mathrm{P}_1$$

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Growth rate? Number of free variables? Equivalent function?

It is non trivial

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Growth rate? Number of free variables? Equivalent function?

$$f(w) = \mathbf{1}_{\text{isOdd}} \quad \in \mathbb{Z}\text{P}_0$$

Definition (Pumpable function)

A function $f\colon \Sigma^\star \to \mathbb{Z}$ is k-pumpable whenever there exists $\alpha_0, \dots, \alpha_k \in \Sigma^\star$, $w_1, \dots, w_k \in \Sigma^\star$, such that

$$\left| f(\alpha_0 \prod_{i=1}^{k} w_i^{X_i} \alpha_i) \right| = \Omega(|X_1 + \cdots + X_k|^k)$$

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That is, one can observe a growth rate at least k by iterating patterns.

General proof, on an example

$$f := \#\,[\mathrm{isOdd}(x)] - \#\,[\mathrm{isEven}(x)] \quad \in \mathbb{Z}\mathrm{P}_1$$

$a \quad a \quad a \quad a \quad a \quad a \quad a \quad a \quad a$

General proof, on an example

$$f := \#\,[\text{isOdd}(x)] - \#\,[\text{isEven}(x)] \quad \in \mathbb{Z}\text{P}_1$$

	a	a	a	a	a	a	a	a	a
Production	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$

General proof, on an example

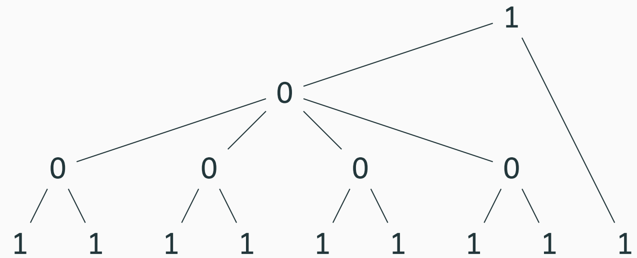
$$f := \#\,[\text{isOdd}(x)] - \#\,[\text{isEven}(x)] \quad \in \mathbb{Z}\text{P}_1$$

$M := (\mathbb{Z}/2\mathbb{Z}, +)$	1	1	1	1	1	1	1	1	1
	a	a	a	a	a	a	a	a	a
Production	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$

$$f := \#[\text{isOdd}(x)] - \#[\text{isEven}(x)] \quad \in \mathbb{Z}\text{P}_1$$

Factorisation [Sim90]

$M := (\mathbb{Z}/2\mathbb{Z}, +)$



$a \quad a \quad a \quad a \quad a \quad a \quad a \quad a \quad a$

Production $\quad +1 \quad -1 \quad +1 \quad -1 \quad +1 \quad -1 \quad +1 \quad -1 \quad +1$

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Skeletons

Factorisation [Sim90]

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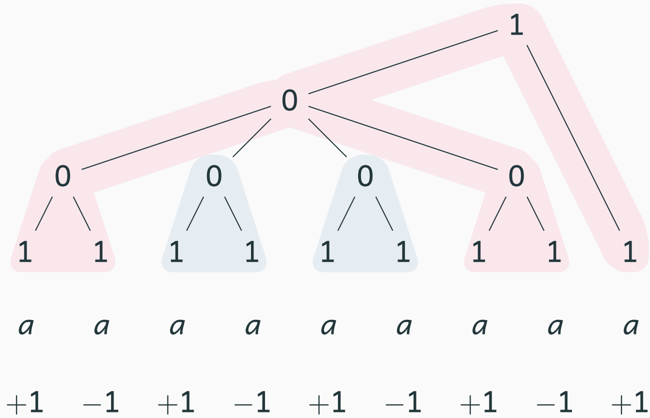
Production



Theorem

The following are equivalent for functions $f\colon \Sigma^\star \to \mathbb{Z}$ and $k \in \mathbb{N}$:

1. $f \in \mathbb{Z}\mathsf{P}_k$.
2. *f is the post-composition of a polyregular function of growth rate k with the $\sum$ operator.*
3. *f is in the closure of regular languages under $\otimes$, $+$, and $z_i \cdot \Box$ and f has growth rate k.*
4. *f is a rational series and f is **not** $k+1$ pumpable.*
5. *f is computed by a weighted automata of ambiguity $\mathcal{O}(|w|^k)$.*

Every conversion is effective.

Proofs?

Deciding aperiodicity

Residuals!

Definition (Residuals of a function f)

$$f(u-)\colon w \mapsto f(uw)$$

$$\mathrm{Res}(f) := \{f(u-) \mid u \in \Sigma^\star\}$$

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If $f \in \mathbb{Z}\mathrm{P}_k$, then $\mathrm{Res}(f)/\mathbb{Z}\mathrm{P}_{k-1}$ is finite!

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"f is a deterministic transducer **up to lower degree errors**"

Residual transducer on an example

$$f(w) := (-1)^{|w|} \times |w| \quad \in \mathbb{Z}\mathsf{P}_1$$

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Residuals up to constant growth

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- $f(a-)$?

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- $f(aa-)$?

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- $f(aa-)$? $g := f(aaw) - f(w) = 2 \times (-1)^{|w|}$

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- And we have exhausted equivalence classes.

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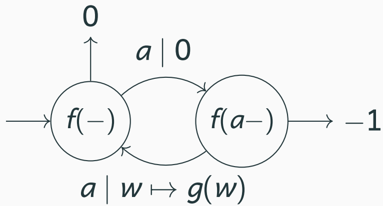

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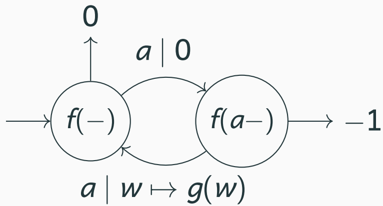

$$\begin{aligned} f(aaaa) &= 0 + f(a \mid aaa) = 0 + g(aa) + f(aa) \\ &= 0 + g(aa) + 0 + f(a \mid a) \\ &= 0 + g(aa) + 0 + g(\varepsilon) + f(\varepsilon) \\ &= 0 + 2 + 0 + 2 + 0 = 4 \end{aligned}$$

Definition (Ultimately polynomial function)

A function $f\colon \Sigma^\star \to \mathbb{Z}$ is ultimately N-polynomial whenever for all $k \in \mathbb{N}$, $\alpha_0, \ldots, \alpha_k \in \Sigma^\star$, $w_1, \ldots, w_k \in \Sigma^\star$, there exists $P \in \mathbb{Q}[X_1, \ldots, X_k]$ such that for large enough $X_1, \ldots, X_k$,

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All $\mathbb{Z}$-polyregular functions are ultimately N-polynomial.

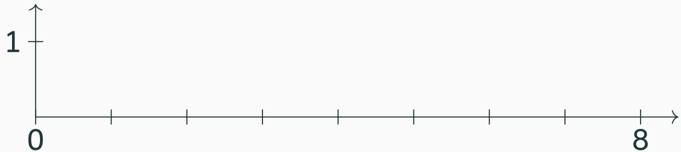
Definition (Ultimately polynomial function)

A function $f\colon \Sigma^\star \to \mathbb{Z}$ is ultimately N-polynomial whenever for all $k \in \mathbb{N}$, $\alpha_0, \ldots, \alpha_k \in \Sigma^\star$, $w_1, \ldots, w_k \in \Sigma^\star$, there exists $P \in \mathbb{Q}[X_1, \ldots, X_k]$ such that for large enough $X_1, \ldots, X_k$,

$$f(\alpha_0 \prod_{i=1}^{k} w_i^{NX_i} \alpha_i) = P(X_1, \ldots, X_k)$$

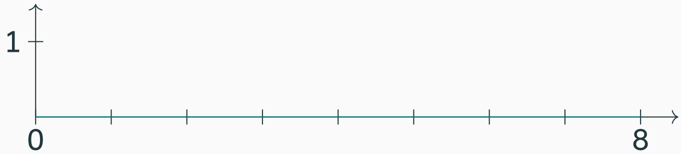
All $\mathbb{Z}$-polyregular functions are ultimately N-polynomial. Star free $\mathbb{Z}$-polyregular functions are ultimately 1-polynomial!

Star-free ...graphically



Star-free ...graphically

$\mathbf{1}_{\text{isOdd}}((aa)^X)$



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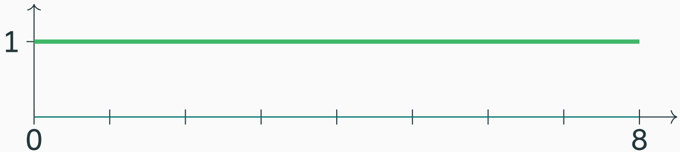
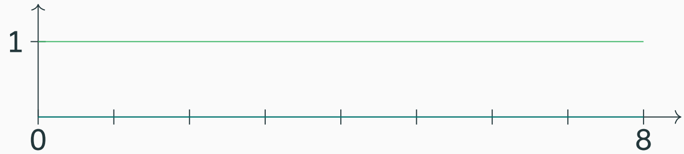
Star-free …graphically



$\mathbf{1}_{\mathsf{isOdd}}((aa)^X)$ $\mathbf{1}_{\mathsf{isOdd}}(a(aa)^X)$

Star-free ...graphically



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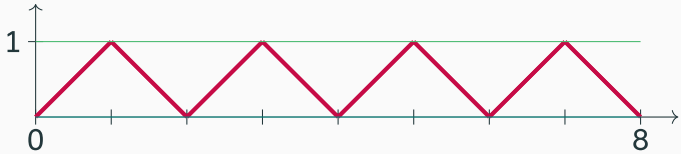

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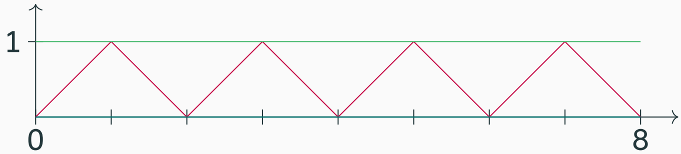

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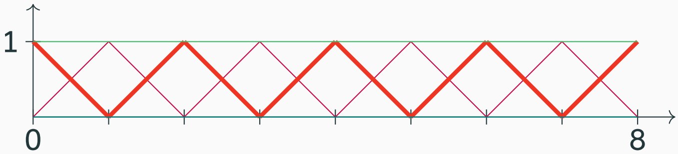
Star-free ...graphically



$\mathbf{1}_{\mathrm{isOdd}}(a^X)$	$\mathbf{1}_{\mathrm{isOdd}}(aa^X)$
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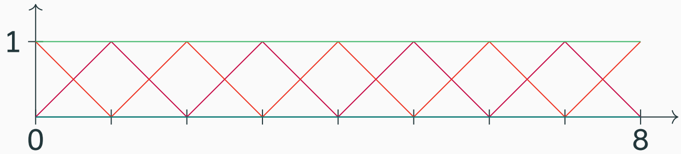
Star-free ...graphically



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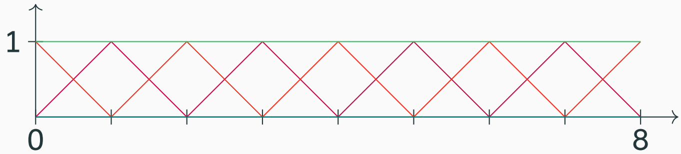

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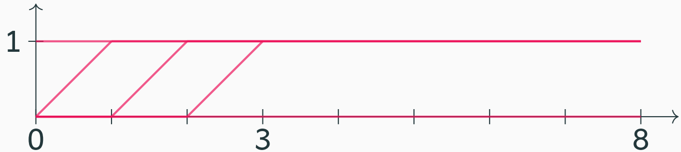

$\mathbf{1}_{|w|_a \geq 3}$

Characterisations

Theorem

The following are equivalent for a $\mathbb{Z}$-rational series f

1. $f \in \mathbb{Z}\mathsf{SF}$.
2. *f is the post-composition of a star-free polyregular function with the $\sum$ operator.*
3. *f is in the closure of star-free languages under $\otimes$, $+$, and $z_i \cdot \square$.*
4. *f is ultimately 1-polynomial (with $k = 1$).*
5. *Minimal representations of f have eigenvalues in $\{0, 1\}$.*
6. *The residual transducer of f is counter-free.*

Every conversion is effective.

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Every conversion is effective.

Furthermore, $\mathbb{Z}\mathsf{SF}_k = \mathbb{Z}\mathsf{SF} \cap \mathbb{Z}\mathsf{P}_k$!

Beyond $\mathbb{Z}$?

Outlook and future work

Outlook

Open questions

- Deciding aperiodicity for $\mathbb{N}$-polyregular functions? (based on ideas from [CGM22])
- Deciding $\mathbb{N}$-polyregular inside $\mathbb{Z}$-polyregular? (note that [Kar77] is not true)
- $\mathbb{N}\mathsf{SF} = \mathbb{N}\mathsf{P} \cap \mathbb{Z}\mathsf{SF}$?
- Defining aperiodicity for $\mathbb{Z}$-rational series in general? (with eigenvalues)

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- Deciding aperiodicity for $\mathbb{N}$-polyregular functions? (based on ideas from [CGM22])
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- Defining aperiodicity for $\mathbb{Z}$-rational series in general? (with eigenvalues)

Slightly related question

Decide if a class of graphs with bounded linear clique-width is well-quasi-ordered?

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