Locality and Preservation Theorems

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Logic. Logic? Logic!

Semantics of FO

Sentences as subsets

$[\![\varphi]\!] \triangleq \{A \mid A \models \varphi\}$

Sentences as subsets

$[\![\varphi]\!] \triangleq \{A \mid A \models \varphi\}$ Wait... where does A live?

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Choices of ambiant space X

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Choices of ambiant space X

M.T. $\mathrm{Struct}(\sigma)$ Stone Spaces, Compactness

F.A.M.T. $\mathrm{Fin}(\sigma)$ No compactness, most model theory fails

Logic. Logic? Logic!

Locality

Local Neighbourhood

$$\mathcal{N}_A(\vec{a}, r) \triangleq \{c \in A \mid d_A(\vec{a}, c) \leq r\}$$

Parent	**Child**
Frank	Bob
Bob	Jeanne
Alice	Martin
Martin	Chloé

Person	**Activity**
Bob	Kite Surf
Jeanne	Rock
Jeanne	Piano
Alice	Kite Surf

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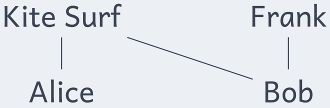

Local Formulas

$$A, \vec{a} \models \varphi(\vec{x}) \text{ if and only if } \mathcal{N}_A(\vec{a}, r), \vec{a} \models \varphi(\vec{x})$$

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- Quantifier free formulas are 0-local.
- $\forall y. E(x, y) \implies x = y$.
- $Q_1 y_1 \in \mathcal{N}(\vec{x}, r). \dots Q_n y_n \in \mathcal{N}(\vec{x}, r). \psi$.

Local sentences?

Name	Syntactic Form
Existential sentence	$\exists \vec{x}. \psi_{\mathsf{qf}}(\vec{x})$
Existential local sentence	$\exists \vec{x}. \psi_{\mathsf{loc}}(\vec{x})$
Basic local sentence	$\exists \vec{x}. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_{i=1}^{n} \psi_{\mathsf{loc}}(x_i)$

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Gaifman Locality Theorem: $\mathsf{FO} = \mathcal{B}(\mathsf{BasicLocal})$.

Independent of X.

Preservation under extensions

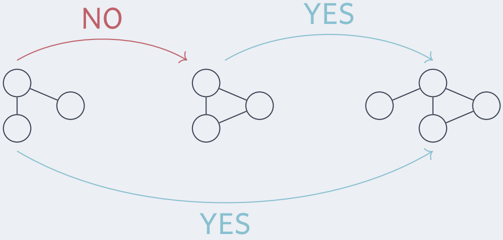
Induced Substructure

Notion of sub-database

Deleting a user from a database should remove *all* entries where that user appears.

$$D_1 \subseteq_i D_2$$

Conversely, a "larger database" will not contain new relations between pre-existing atoms.



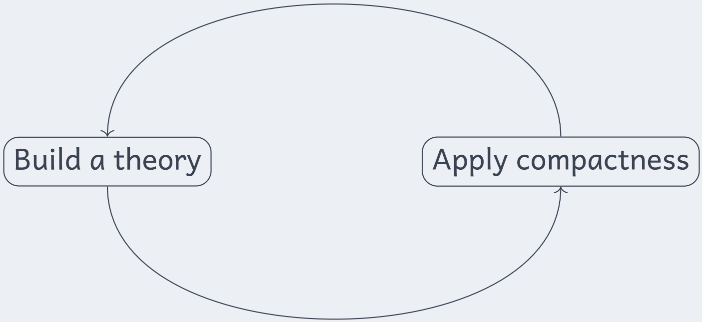
What are the FO sentences that are invariant under *database extensions*?

Answer: existential sentences Łoś [2], Tarski [4].

Preservation under extensions

Proof Scheme

Preservation under extensions



This terminates by compactness.

Mitigated results

- Tait [3]: The theorem fails on $\mathsf{Fin}(\sigma)$.
- Atserias, Dawar, and Grohe [1]: The theorem succeeds on bounded degree structures

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- Tait [3]: The theorem fails on $\mathrm{Fin}(\sigma)$.
- Atserias, Dawar, and Grohe [1]: The theorem succeeds on bounded degree structures
- Under extra assumptions: hereditary and closed under disjoint unions

Proof scheme over structures of bounded degree.

1. Use the Gaifman Normal Form.
2. Consider a minimal structure $A \models \varphi$.
3. Extract "positive witnesses" from A to build $A_0 \subseteq_i A$.
4. If $A_0 \not\models \varphi$, Repeat step (3) with $A \subseteq_i A_0 \uplus A$, finding new witnesses in A far from A_0.

This process terminates in a number of steps dependent on φ but not A.

Disjoint unions... Sort of

Local Elementary Embeddings

Can we extract a pattern?

Induced Substructure

Preserve quantifier free formulas.

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Preserve local formulas.

Over finite structure: disjoint unions!

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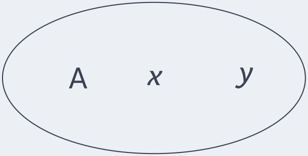

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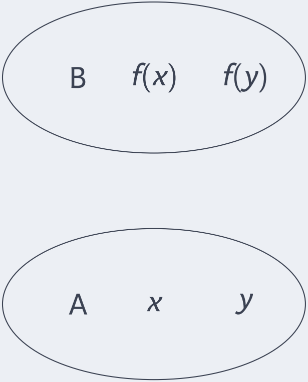

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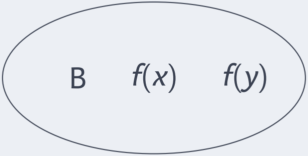

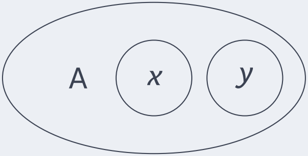

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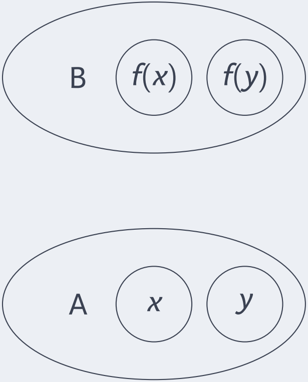

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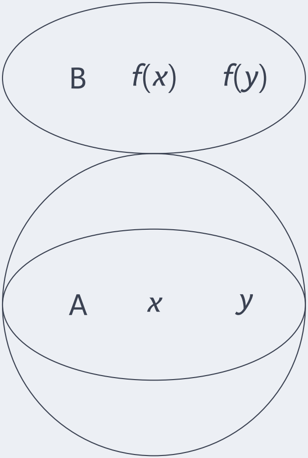

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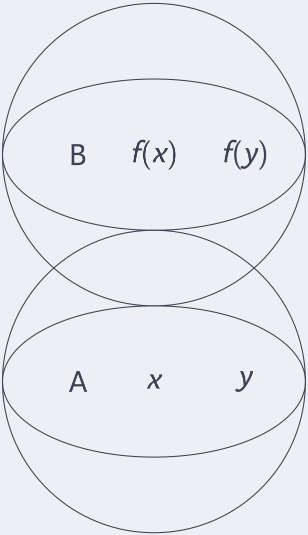
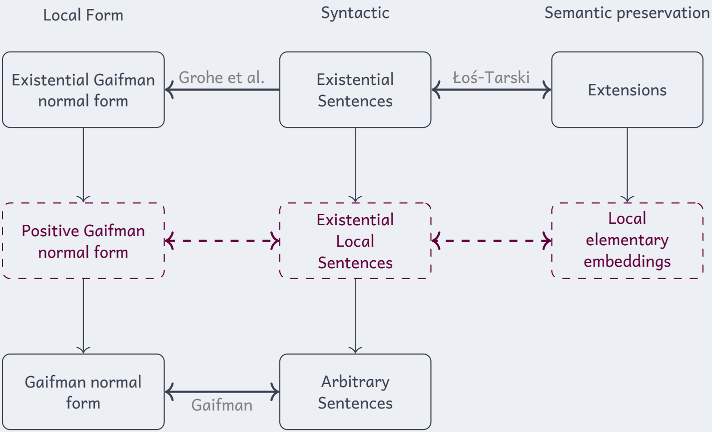


Disjoint unions… Sort of

A nice preservation theorem


Local Form
Syntactic
Semantic preservation
Existential Gaifman normal form
Grohe et al.
Existential Sentences
Łoś-Tarski
Extensions
Positive Gaifman normal form
Existential Local Sentences
Local elementary embeddings
Gaifman normal form
Gaifman
Arbitrary Sentences

Profit?

There exists φ preserved under $\uplus$ and not existential local.

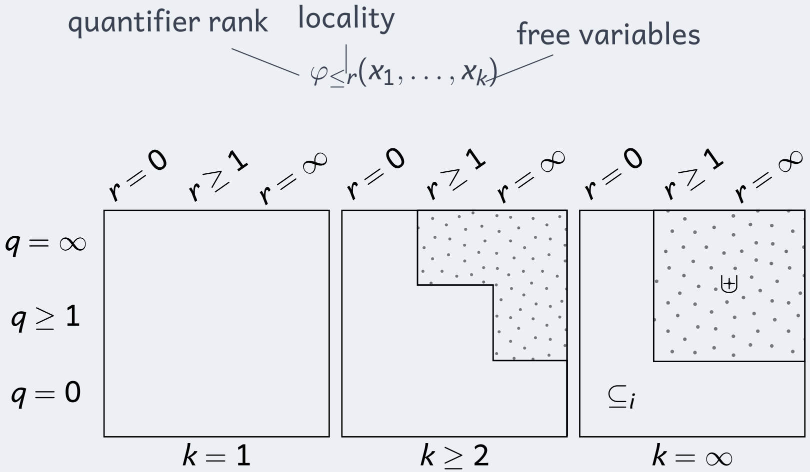
Undecidable/Uncomputable problems

- Decide if *a* sentence is preserved under disjoint unions
- Decide if *a* sentence is equivalent to *an* existential local sentence
- Compute, under the promise that the sentence is equivalent to *an* existential local sentence, *an* equivalent existential local sentence.

But what about Atserias et al.'s proof scheme?

Specific implications

Weakening the preorders



Did we actually gain anything?

YES

Factorised Proof Scheme

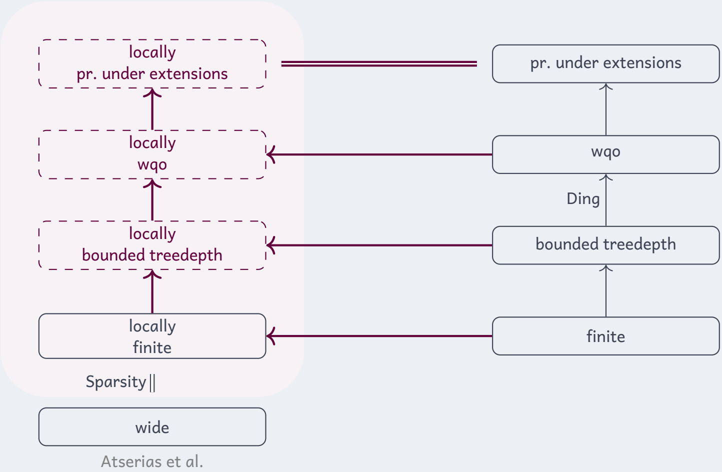
sentence $\rightarrow$ existential local $\rightarrow$ existential

Theorem

For a hereditary class X closed under disjoint unions the following are equivalent

1. *Preservation under extensions holds,*
2. *Preservation under extensions holds over* $\mathrm{Balls}(X, r, k)$.

New classes obtained through locality $X \subseteq \mathrm{Fin}(\sigma)$



Thank You

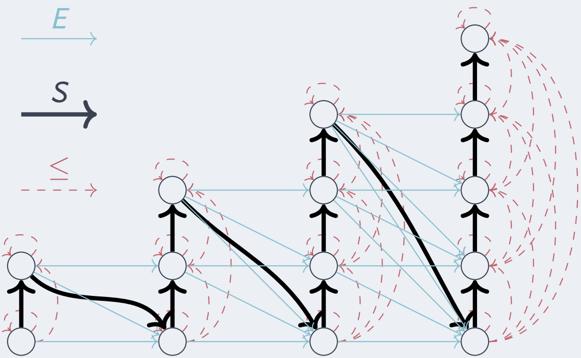
[1] Atserias, A., Dawar, A., and Grohe, M. (2008). Preservation under extensions on well-behaved finite structures. *SIAM Journal on Computing*, 38(4):1364–1381.

[2] Łoś, J. (1955). On the extending of models (I). *Fundamenta Mathematicae*, 42(1):38–54.

[3] Tait, W. W. (1959). A counterexample to a conjecture of Scott and Suppes. *Journal of Symbolic Logic*, 24(1):15–16.

[4] Tarski, A. (1954). Contributions to the theory of models. I. *Indagationes Mathematicae (Proceedings)*, 57:572–581.

A Very Grid-Like Structure



The class of such structures is definable using the negation of an existential local sentence.

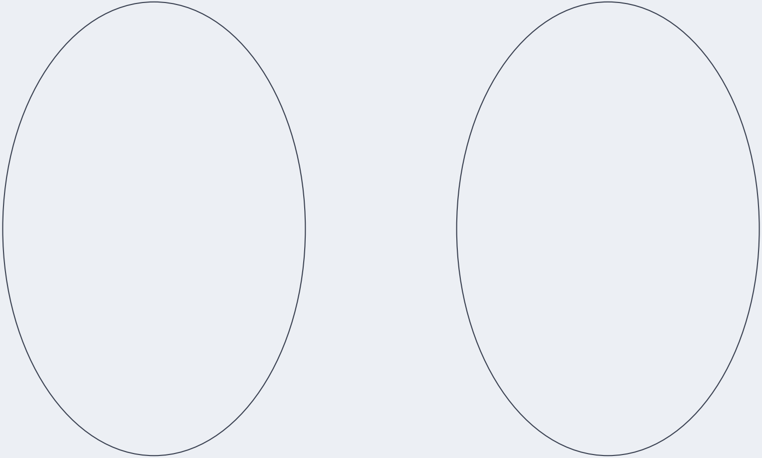
Logical Cores and Local Elementary Embeddings

Lemma (Type covering)

For all $r, q, k \geq 0$, there exists K_m and R_m such that one can build for every structure A

- C^A *exhausting "rare types".*
- G^A *collecting "frequent types".*
- *Controlled size and distances independently from A.*

Logical Cores and Local Elementary Embeddings (2)



$A \models \varphi$ $\qquad A \to_q^{r,k} B$ $\qquad B$

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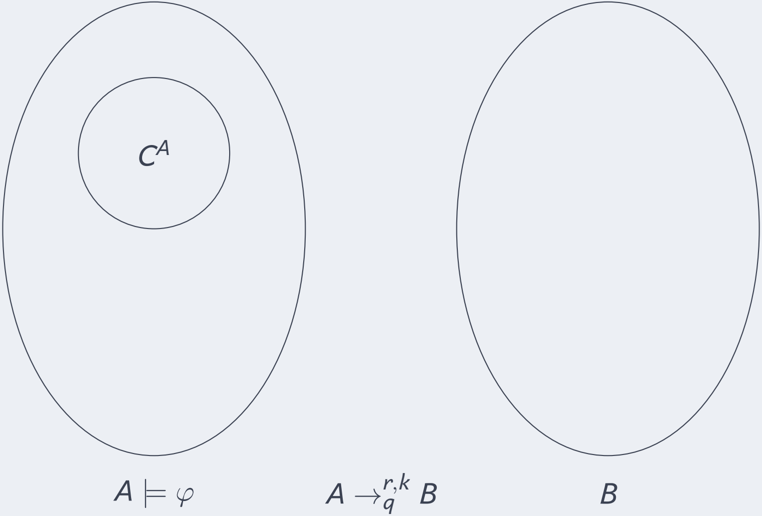

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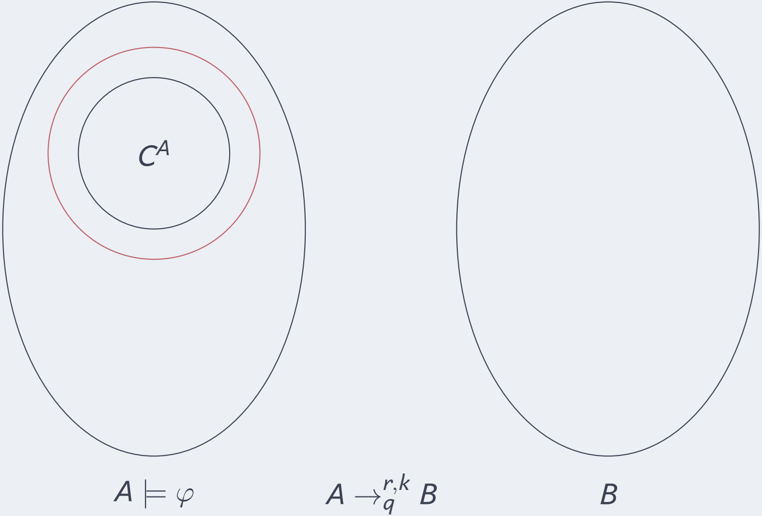

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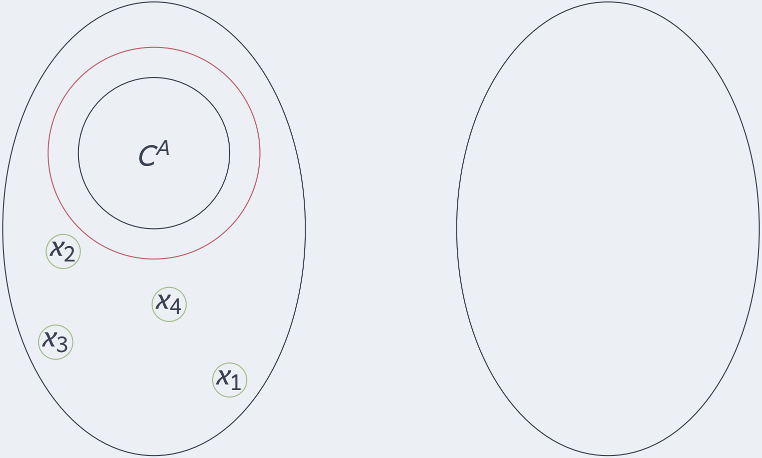

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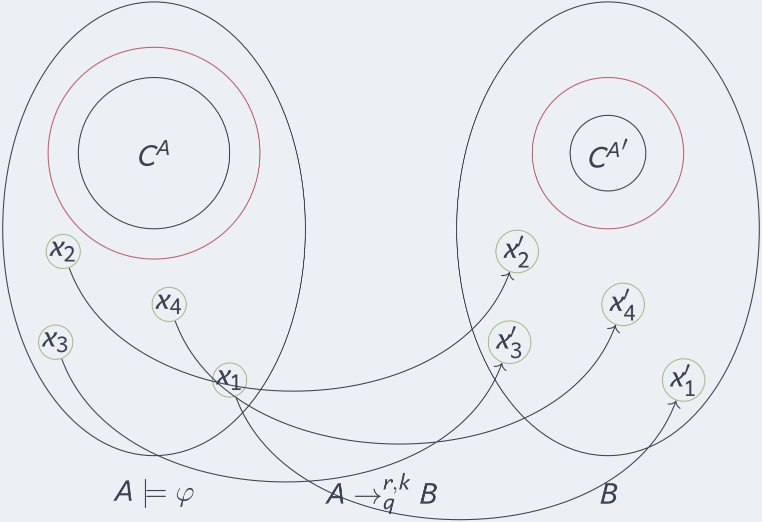

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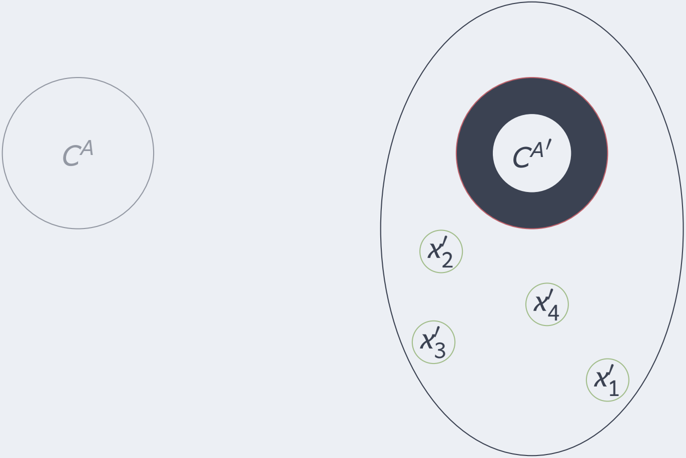

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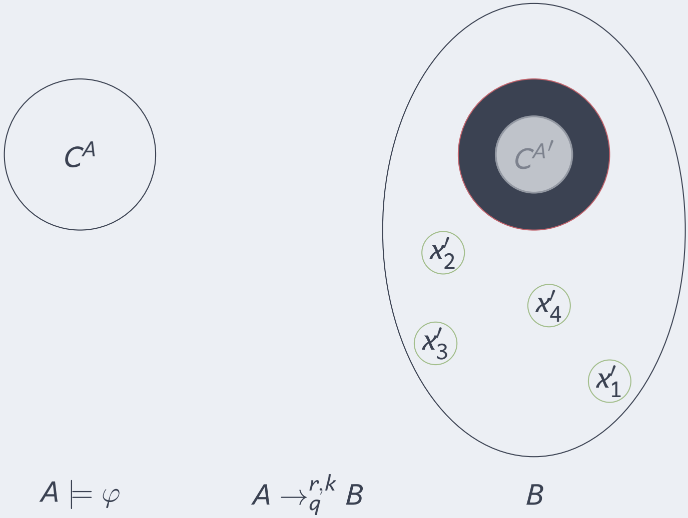

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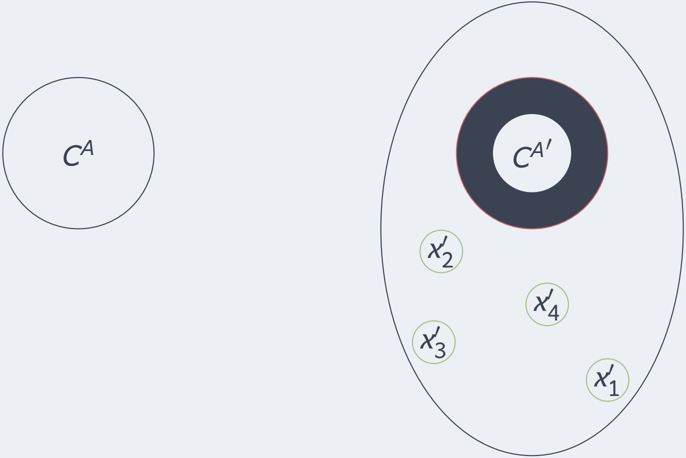

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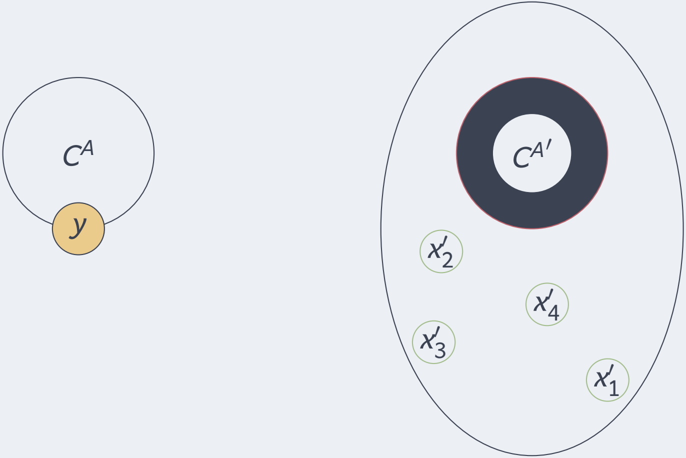

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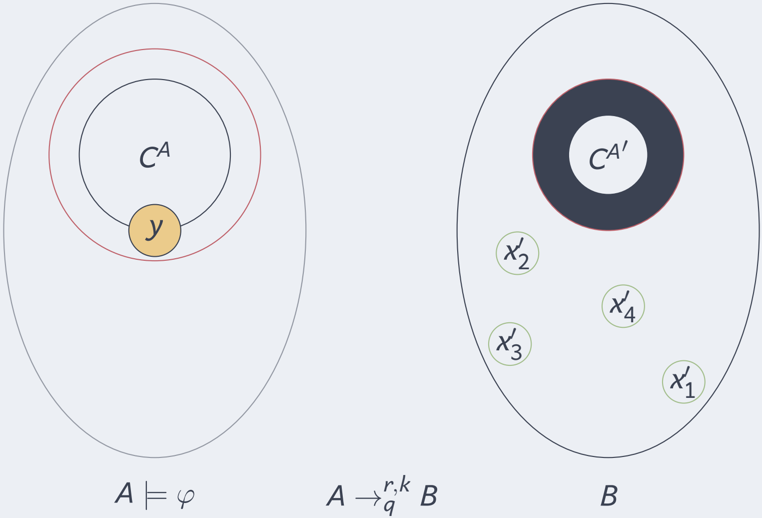

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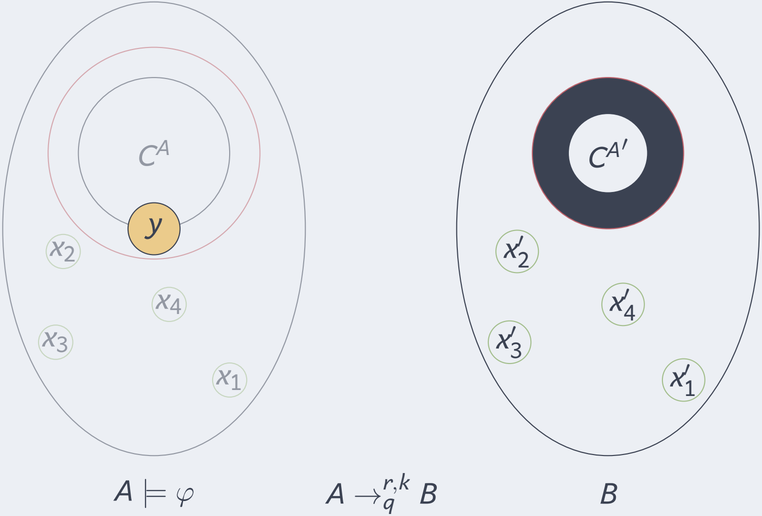

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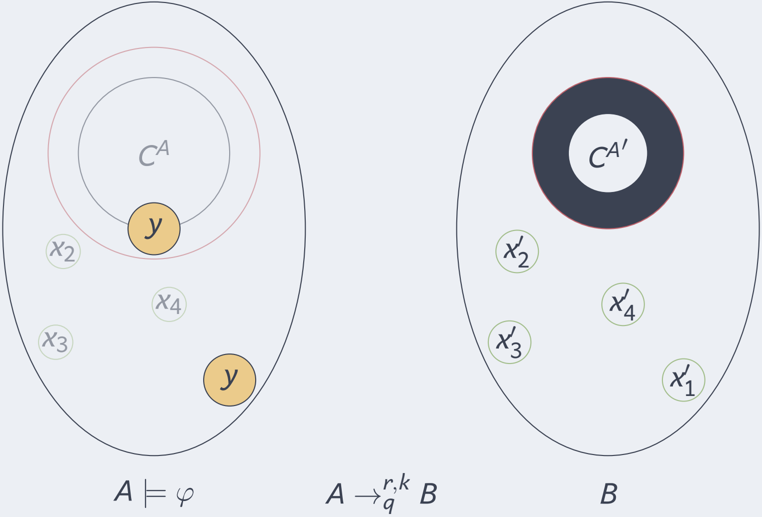

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