Generic Noetherian Theorems

Defining Noetherian Topologies Through Iterations

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1. WQOs and Beyond
2. Minimal Bad Sequence Arguments
3. Applications
4. Open Questions

1. WQOs and Beyond

   Well-quasi-orders

   Noetherian Spaces

2. Minimal Bad Sequence Arguments

3. Applications

4. Open Questions

WQOs and Beyond

Well-quasi-orders

Think Fast!

Quasi-Order	Sequence	Good
$(\mathbb{N}, =)$	$i \mapsto i$	✗
$(\mathbb{N}, \leq)$	$i \mapsto i$	✓
$(\{a, b\}, \sqsubseteq)$	$i \mapsto a^i$	✓
$(\{a, b\}, \sqsubseteq)$	$i \mapsto ba^i$	✗
$(\{a, b\}, \leq_*)$	$i \mapsto ba^i$	✓
$(\mathcal{G}, \subseteq_i)$	$i \mapsto C_i$	✗
$(\mathcal{G}, \leq_{minor})$	$i \mapsto C_i$	✓

Algebra of WQOs

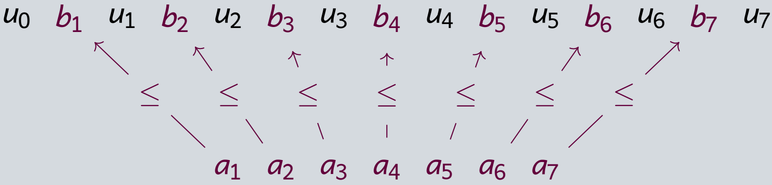
$D ::= (\mathbb{F}, =)$	finite set
$\mid (\mathbb{N}, \leq)$	natural numbers
$\mid \Sigma_{i=1}^{n} D_i$	finite disjoint sums
$\mid \Pi_{i=1}^{n} D_i$	finite products
$\mid D^*$	finite words, subword embedding
$\mid D^{\circledast}$	finite multisets, multiset embedding
$\mid \mathcal{P}_f(D)$	finite sets, Hoare embedding
$\mid T(D)$	finite trees, Kruskal embedding

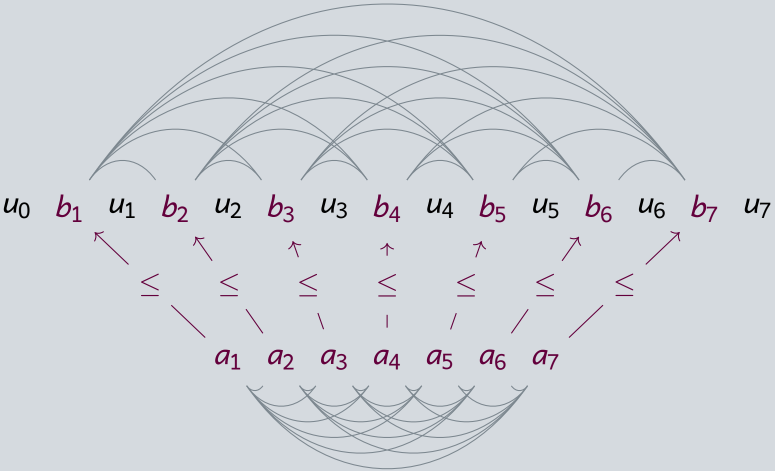
Keep in mind

How do we choose the order?

- → Lattice of WQOs
- → Not complete

Ordering on Words: suffix, pointwise & substructures

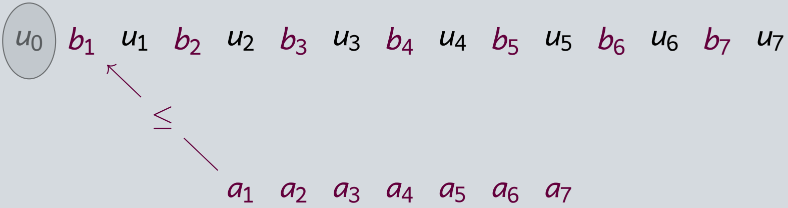




u_0 b_1 u_1 b_2 u_2 b_3 u_3 b_4 u_4 b_5 u_5 b_6 u_6 b_7 u_7
≤ ≤ ≤ ≤ ≤ ≤ ≤
a_1 a_2 a_3 a_4 a_5 a_6 a_7

$$u_0 \quad b_1 \quad u_1 \quad b_2 \quad u_2 \quad b_3 \quad u_3 \quad b_4 \quad u_4 \quad b_5 \quad u_5 \quad b_6 \quad u_6 \quad b_7 \quad u_7$$

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7$$


u_0 b_1 u_1 b_2 u_2 b_3 u_3 b_4 u_4 b_5 u_5 b_6 u_6 b_7 u_7
$\leq$
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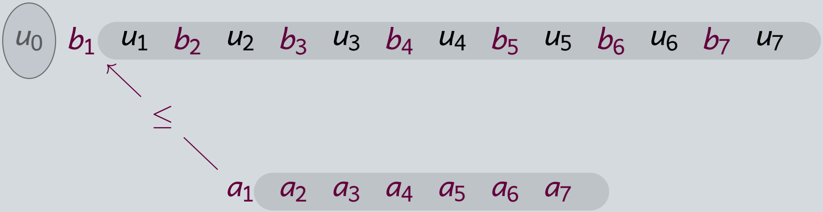
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Failures

▲ Some constructions fail to preserve WQOs

X^*	prefix ✗, suffix ✗, factor ✗, pointwise ✗, subword ✓
X^ω	infinite words with subword ✗
$\mathcal{P}(X)$	powerset with embedding ✗

◆ The Powerset Problem

Rado's structure Rado (1954)

WQOs and Beyond

Noetherian Spaces

Topology Cheatsheet

Pre-order $\leq$	**Topology** $\mathsf{Alex}(\leq)$
U is upwards-closed	U is open
f is monotone	f is continuous
E has finitely many minimal elements	E is compact
wqo	Noetherian

Better than posets?

Recap of Jean's talk

- Posets are topological spaces with the Alexandroff topology

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◆ **Recap of Jean's talk**

- Posets are topological spaces with the Alexandroff topology
- There are topological equivalent to the constructions on posets (products, sums, ...)
- wqo is a way of stating compactness

▲ **Different approach than BQO**

We broaden the notion of "well-behaved" space.

Noetherian space

A topological space (X, τ) is *Noetherian* if every subset of X is compact.

Space	Topology	Compact	Noetherian	WQO
$\mathbb{N}$	Alex($\leq$)	✓	✓	✓
$\mathbb{N}$	cofinite	✓	✓	✗
$\mathbb{N}$	discrete	✗	✗	✗
Σ^*	Upper($\sqsubseteq$)	✓	✓	✗
Σ^*	Alex($\sqsubseteq$)	✓	✗	✗
Σ^*	Alex($\leq_*$)	✓	✓	✓
$\mathbb{R}$	metric	✗	✗	✗
$[0, 1]$	metric	✓	✗	✗
$\mathbb{C}$	Zariski	✓	✓	✗
Σ^*	regular subword	✓	✓	"✓"

Algebra of Noetherian Spaces

$D ::= (X, \mathrm{Alex}(\leq))$	$(X, \leq)$ wqo
$\mid \Sigma_{i=1}^{n} D_i$	finite disjoint sums
$\mid \Pi_{i=1}^{n} D_i$	finite products
$\mid D^*$	finite words, regular subword topology
$\mid D^{\circledast}$	finite multisets, multiset topology
$\mid \mathcal{P}(D)$	arbitrary subsets, lower Vietoris topology
$\mid T(D)$	finite trees, regular subtree topology
$\mid \mathcal{S}(D)$	sobrification
$\mid X^{\omega}$	infinite words, regular subword topology

Focus on Finite Words

▲ **Orders on Finite Words**

- prefix ✗

▲ **Topologies on Finite Words**

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- suffix ✗

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- prefix topology ✗
- $\sum_{k \geq 0} X^k$ ✗
- regular subword topology $[U_1, \ldots, U_n]$ Goubault-Larrecq (2013) ✓

◆ Intuition

topology $\simeq$ logic with infinite disjunction

- **Intuition**

topology $\simeq$ logic with infinite disjunction

- **Basic LTL Formulas** PLTL

$$P_U \text{ for } U \text{ open}$$
$$\Diamond\varphi \text{ for } \varphi \text{ open}$$

Regular subword topology $= \langle \Diamond\varphi \mid \varphi \in \text{PLTL} \rangle$

Keep in mind

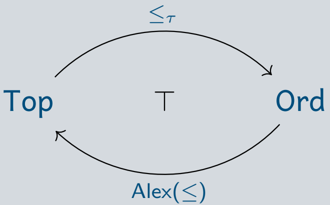
How do we choose the ~~order~~ topology?

◆ Getting back to preorders

Assume we know the preorder we want. Assert that the topology "corresponds".

Definition (Specialisation Preorder)

$$a \leq_\tau b \iff \forall U \in \tau, a \in U \implies b \in U.$$



How do you prove...

The following are WQOs

- finite words X^* with the Higman's word embedding
- finite trees $T(X)$ with Kruskal's tree embedding
- finite 2-trees $\mu Y.X \times T(Y)$
- finite n-trees...
- Graphs generated by a totally ordered monoid with induced subgraph (Daligault et al., 2010)

Using a minimal bad sequence argument.

Minimal Bad Sequence Arguments

For WQOs

- Consider a bad sequence $(w_n)_{n\in\mathbb{N}}$ that is minimal for $\sqsubseteq$
- It cannot contain ε
- Hence $w_n = a_n v_n$
- The set $S \triangleq \{v_n \mid n \in \mathbb{N}\}$ is a wqo
- Hence $X \times S$ is a wqo
- But $\{w_n \mid n \in \mathbb{N}\}$ reflects in $X \times S$

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- Hence $w_n = a_n v_n$ — decompose using $F(Y) = 1 + X \times Y$.
- The set $S \triangleq \{v_n \mid n \in \mathbb{N}\}$ is a wqo — minimality
- Hence $X \times S$ is a wqo — stability through $F(Y)$
- But $\{w_n \mid n \in \mathbb{N}\}$ reflects in $X \times S$ — recompose via F

The proof in full generality

▲ Sufficient conditions?

- Studied by Hasegawa (2002) and Freund (2020)
- Lots of category theory

◆ Basic idea: order on inductive constructions

- Finite words: $\mu Y.1 + X \times Y$
- Finite trees: $\mu Y.X \times Y^*$
- Finite 2-trees: $\mu Y.X \times T(Y)$
- ...

What is the connection with our question?

It answers our question on wqos!

And provides *a* “canonical” ordering on the fixed points.

◆ Goals of this talk

- Adapt to *a* topological setting
- Justify existing constructions
- Provide new Noetherian spaces

Minimal Bad Sequence Arguments

Refinement Functions

Strategy over Topological Spaces

◆ Idea: *avoid categories*

- The property of being Noetherian does not depend on the points
- We can iteratively refine the topology

▲ Consequence

We decouple the construction of the space and of its topology.

Refinement Function

Fix X a set, a refinement function F maps topologies over X to topologies over X and

- Preserve Noetherian topologies
- Is monotone: $\tau \subseteq \tau' \implies F(\tau) \subseteq F(\tau')$

▲ Limit topology

One can iterate transfinitely F. One can define its least fixed point.

Iterating does not always work out

▲ Building the prefix topology

$F(\tau) \triangleq \langle \{ T \cdot V \mid T \in \theta, V \in \tau \} \rangle$

$\bigcup_{i \in \mathbb{N}} a^i b \Sigma^*$ does not stabilise

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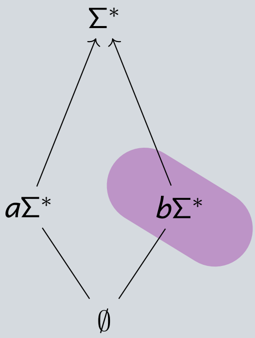

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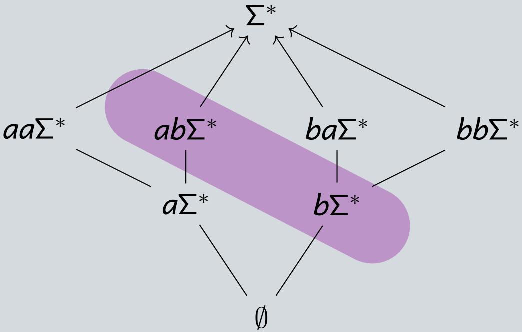

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What went wrong?

◆ **Recall why the subword embedding works**

We lack the notion of substructures. For $\mu Y.1 + X \times Y$, we have to consider suffixes.

What is a topology expander?

◆ Correcting the prefix topology

$F(\tau) \triangleq \langle \{ \uparrow_{\sqsubseteq} T \cdot V \mid T \in \theta, V \in \tau \} \rangle$

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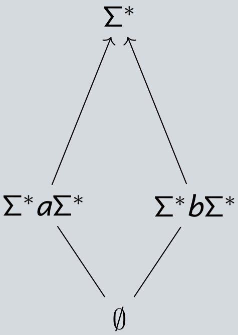

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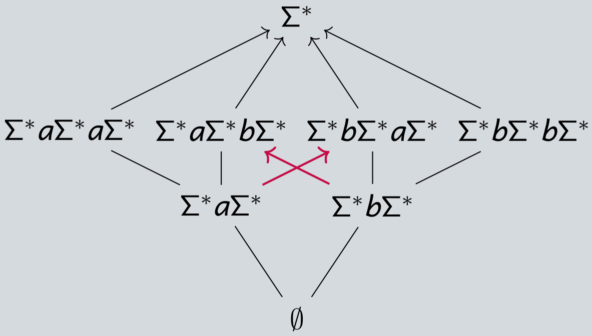
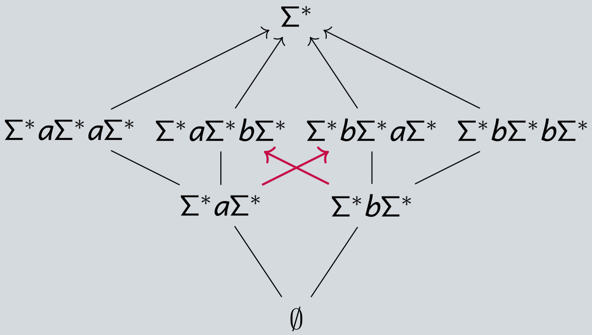


◆ Correcting the prefix topology

$F(\tau) \triangleq \langle \{ \uparrow_{\sqsubseteq} T \cdot V \mid T \in \theta, V \in \tau \} \rangle$ lfp F is the regular subword topology.



Minimal Bad Sequence Arguments

Topology Expanders

Can we replace "suffix" in general?

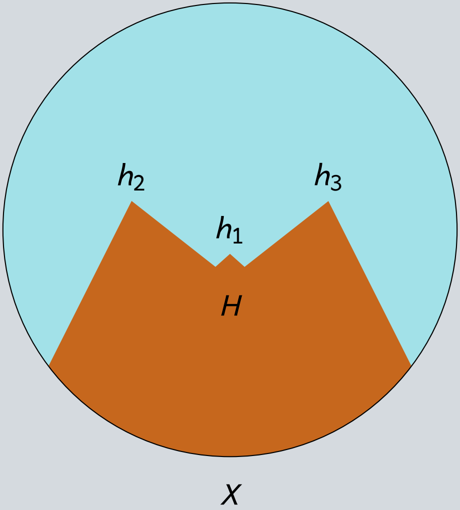
Definition

$\tau|H \triangleq \{U \cup H^c \mid U \in \tau\}$

◆ **In the case of $\tau = \mathrm{Alex}(\leq)$ and H downwards closed**

$$
\begin{aligned}
x \leq_{\tau|H} y &\iff \forall U \in \tau|H, x \in U \Rightarrow y \in U \\
&\iff \forall U \in \tau, x \in U \cup H^c \Rightarrow y \in U \cup H^c \\
&\iff x \in \uparrow x \cup H^c \Rightarrow y \in \uparrow x \cup H^c \\
&\iff \begin{cases} x \leq y \in H \\ y \notin H \end{cases}
\end{aligned}
$$

Can we replace "suffix" in general? YES



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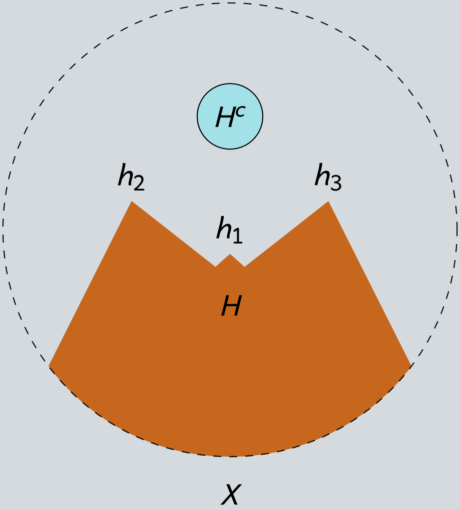


Topology Expander

F is a refinement function and

$$\forall \tau \subseteq F(\tau), \forall H \text{ closed in } \tau, F(\tau)|H \subseteq F(\tau|H)|H$$

Translating and building intuition

The condition on topologies

$F(\tau)|H \subseteq F(\tau|H)|H$

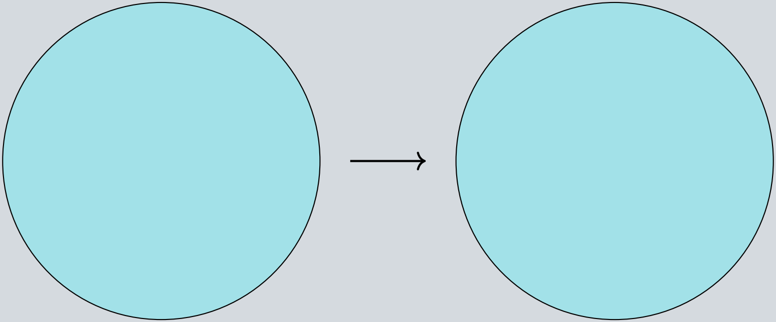
Specialisation preorders

$\leq_{F(\leq)|H} \supseteq \leq_{F(\leq|H)|H}$

▲ **Refinement happens locally in closed sets**

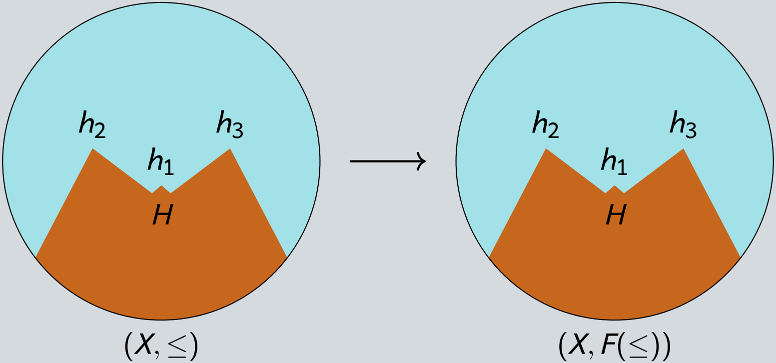
$$x\, F(\leq|H)\, y \in H \implies x\, F(\leq)\, y \in H$$

Refinement happens “locally”



$(X, \leq)$ $\longrightarrow$ $(X, F(\leq))$

Refinement happens "locally"



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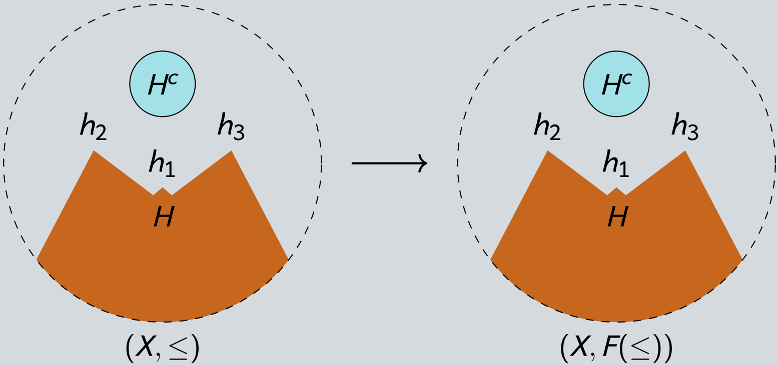

Theorem (Iteration)

If τ is Noetherian and $\tau \subseteq F(\tau)$ then $F^{\alpha}(\tau)$ is Noetherian for all α.

What is a minimal bad sequence?

Definition (Good sequence)

$(U_n)_{n\in\mathbb{N}}$ is good if $\exists i. U_i \subseteq \bigcup_{j<i} U_j$

A sequence that is not good is bad.

Goubault-Larrecq (2013, Lemma 9.7.31)

If τ is generated by B, $\sqsubseteq$ is well-founded on B, and (X, τ) is not Noetherian, then there exists *a* $\sqsubseteq$-minimal bad sequence of opens in B.

Definition (Depth)

The first ordinal β such that $U \in F^\beta(\tau)$.

◆ Very Sketchy Sketch

- If $\alpha = \beta + 1$: automatic

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- $U_n \cup V_n$ open in $F(\mathcal{U})|H_n$ which is Noetherian

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- $U_n \cup V_n$ open in $F(\mathcal{U})|H_n$ which is Noetherian
- $U_{n_0} \subseteq U_{n_0} \cup V_{n_0} \subseteq \bigcup_{n<n_0} U_n \cup V_n = \bigcup_{n<n_0} U_n$ absurd

Applications

Divisibility Topology and Inductive Definitions

Definition (Divisibility topology)

Let $G(\mu G) \simeq_\delta \mu G$ and $\trianglelefteq$ the "substructure" ordering on μG. The divisibility topology is the least fixed point of
$F_\Diamond(\tau) \triangleq \langle \{ \uparrow_\trianglelefteq \delta(U) \mid U \text{ open in } G^T(\mu G, \tau) \} \rangle$

Theorem (Coincidence)

The divisibility topology is Noetherian, and coincides with the Alexandroff topology of the divisibility preorder (Hasegawa, 2002, Def. 2.7) "when it makes sense".

Sanity Check

The topologies over trees and words from Goubault-Larrecq (2013) are the divisibility topologies of the appropriate functors

1. $X^* = \mu Y.1 + X \times Y$
2. $T(X) = \mu Y.X \times Y^*$

▲ **We justified their definition**

Applications

On big spaces

Theorem (Recurrent Subword Topology)

X^α is Noetherian with the topology generated by the following closed sets

$$
\begin{aligned}
P ::= {} & T^? && T \text{ closed} \\
\mid {} & P_1 \dots P_n && \\
\mid {} & P^{<\beta} && \beta \leq \alpha
\end{aligned}
$$

▲ This is clearly not WQO!!!

We took advantage of the refinement on a pre-existing space!

1. WQOs and Beyond

2. Minimal Bad Sequence Arguments

3. Applications

4. Open Questions

A non exhaustive list

◆ **Topological spaces and closed maps**

Can we re-interpret the limits in this setting? What would be the relation with PO-dilators Girard (1981); Freund (2020)?

◆ **Actions on Invariants**

What is the effect of the fixed point on $\leq_\tau$? On the stature of the resulting space?

◆ **Check that we can extend the algebra**

Check that if $G(X, Y_1, \dots Y_n)$ is good, so is $\mu X.G(X, Y_1, \dots, Y_n)$.

Thank You !

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The Official Université Paris-Saclay Color Palette

It is not that bad!