Preservation theorems

At the crossroads of topology and logics

Aliaume Lopez

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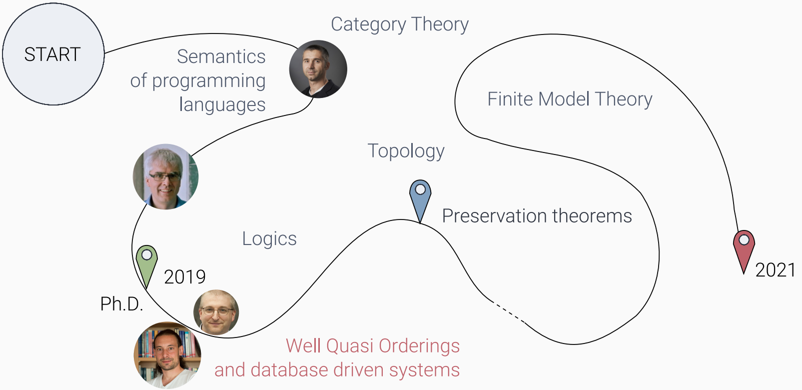
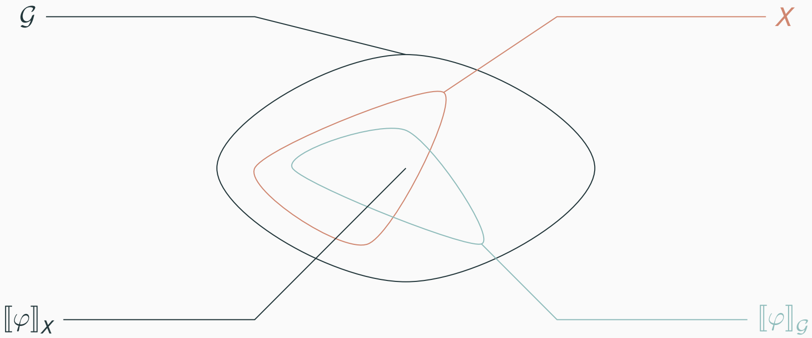
Under the supervision of Sylvain Schmitz and Jean Goubault-Larrecq

Ph.D


START
Semantics
of programming
languages
Category Theory
Finite Model Theory
Topology
Preservation theorems
Logics
2019
Ph.D.
2021
Well Quasi Orderings
and database driven systems

Models of a sentence $\varphi \in \mathrm{FO}[\sigma]$

$$[\![\varphi]\!] : \mathcal{G} \to \{0,1\} \qquad [\![\varphi]\!]_X \triangleq \{G \in X \mid G \models \varphi\}$$



Increasing information ?

$$[\![\varphi]\!] : (\mathcal{G}, \leq) \to (\{0,1\}, \leq_{\mathbb{B}})$$

Monotone sentences over X

- $([\![\varphi]\!])_{|X}$ is *non-decreasing*
- $[\![\varphi]\!]_X = \uparrow [\![\varphi]\!]_X \triangleq \{G \in X \mid \exists H \in [\![\varphi]\!]_X, H \leq G\}$
- For all $(G_1, G_2) \in X^2$ such that $G_1 \leq G_2$ and $G_1 \models \varphi$, $G_2 \models \varphi$.

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Intuition

A query φ that acts on databases respecting some information order $\leq$.

Preservation theorems

Simplify queries

"Monotone sentences are expressible in some fragment F of FO[σ]"

Simplify queries

“Monotone sentences are expressible in some fragment F of FO[σ]”

Struct(σ)	$\leq$	F	Fin(σ)
Łós-Tarski ✓	$\subseteq_i$	EFO	✗ Tait (1959)
Tarski-Lyndon ✓	$\subseteq$	$\text{EPFO}^{\neq}$	✗ Ajtai and Gurevich (1994)
H.P.T. ✓	$\rightarrow$	EPFO	✓ Rossman (2008)

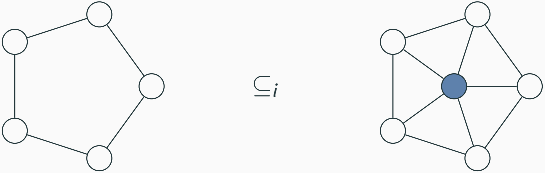
An example of sentence

$$\varphi \triangleq \forall x.\exists y.\neg(xEy) \land x \neq y$$

Dependent on the set X

An example of sentence

$$\varphi \triangleq \forall x . \exists y . \neg(xEy) \wedge x \neq y$$



$\models \varphi$ ✓ $\models \varphi$ ✗

Not Monotone!

Paths $\mathcal{P}$

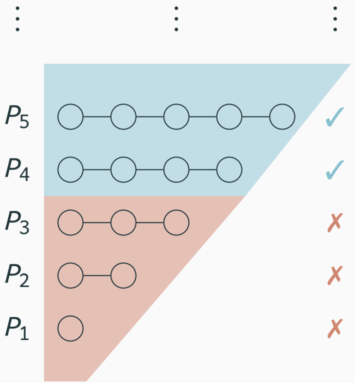
Structure of paths

- Totally ordered for $\subseteq_i$
- The sentence φ is monotone

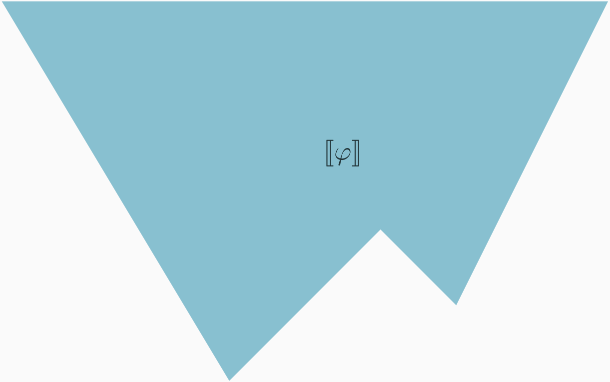
Rewriting φ

$$[\![\varphi]\!]_{\mathcal{P}} = \uparrow\{P_4\}$$

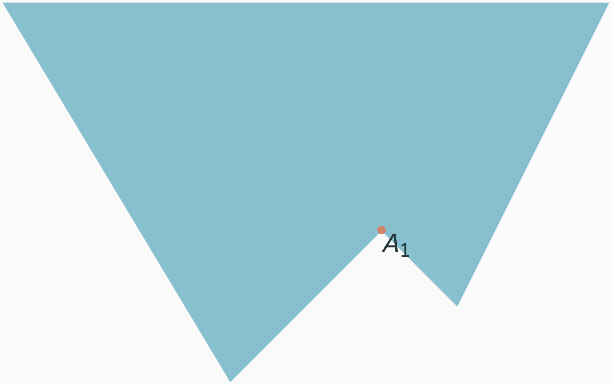
$$\begin{aligned}\varphi \equiv_{\mathcal{P}} & \exists x_1, x_2, x_3, x_4. \\ & x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \\ & x_1 \neq x_4 \wedge x_2 \neq x_3 \wedge \\ & x_2 \neq x_4 \wedge x_3 \neq x_4 \\ \triangleq\ & \psi_4\end{aligned}$$



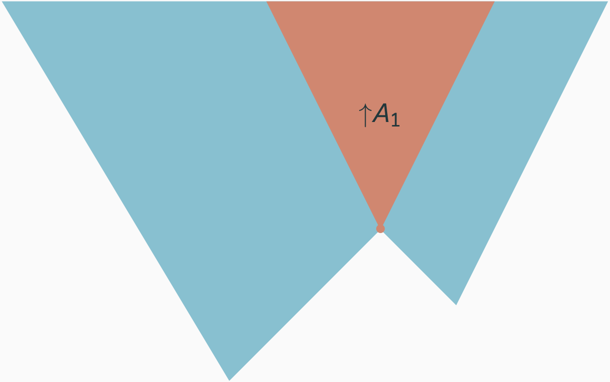
Generalise

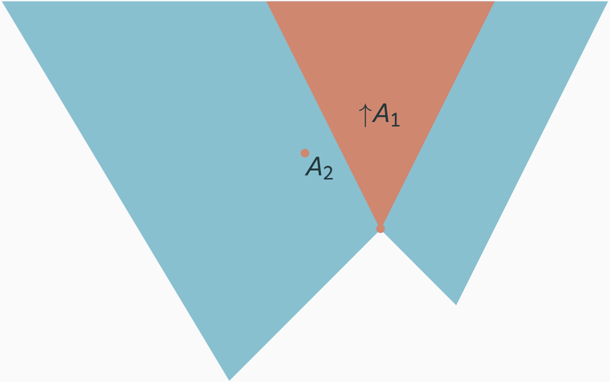


Generalise

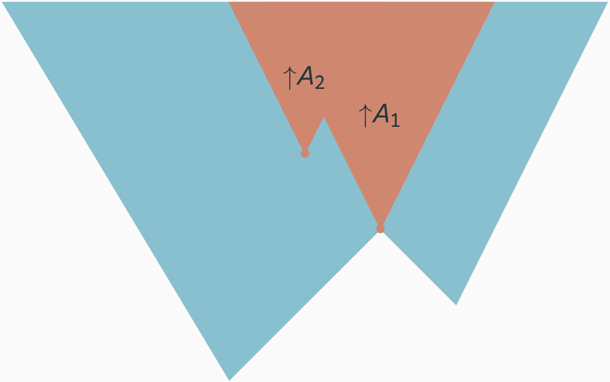


Generalise

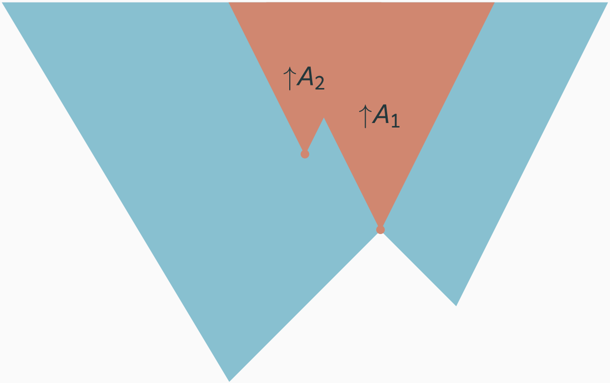




$\uparrow A_1$
A_2

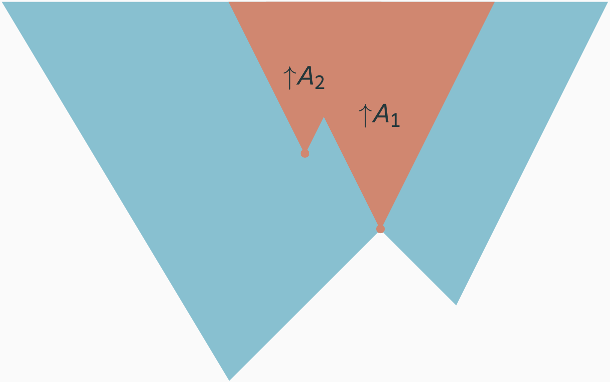

$\uparrow A_2$
$\uparrow A_1$

Generalise



Well-quasi-ordered (Kruskal, 1972) $\implies$ this process terminates

Generalise



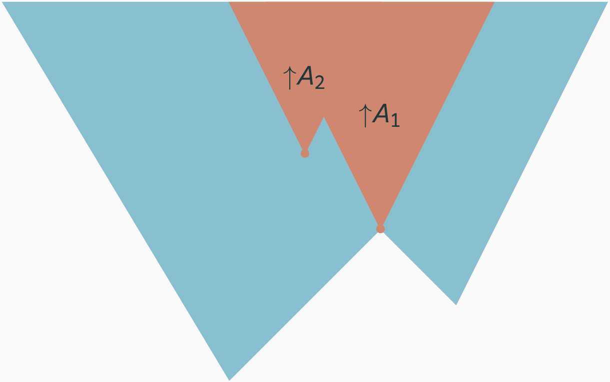
$[\![\varphi]\!]$ is *compact* $\implies$ this process terminates

Generalise

Pre-order	Topology
$\leq$	τ
upwards-closed	open
monotone	continuous
$\uparrow F$, F finite	compact

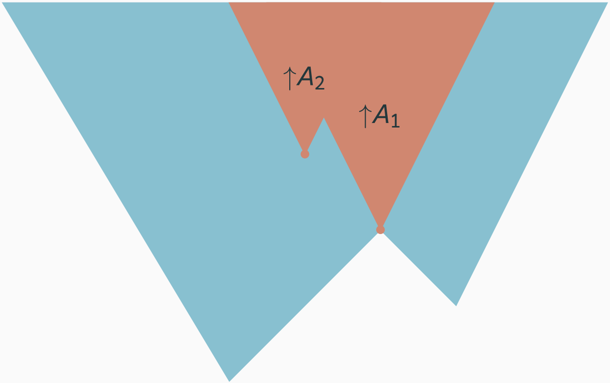
$[\![\varphi]\!]$ is *compact* $\implies$ this process terminates

Generalise



When does this process terminates?

Generalise



When definable open sets are compact

Definition

Logically-presented pre-spectral spaces (lpps)

For a triplet $\langle X, \tau, \mathrm{FO}[\sigma] \rangle$

Pre-spectral τ is generated by a *lattice* of compact open sets.
(see Dickmann et al., 2019)

Presented definable open sets are compact.

Remark

In a lpps, compact open sets are definable.

Example of previous work

Study of preservation theorems

Spaces that are lpps automatically build *preservation theorems*.

Structural properties (Lopez, 2021)

Stability under finite products, sums, definable open/closed subsets ...

Open questions

Connection to *sparsity* of the classes, efficient evaluation of FO ...

Local preservation

What is the semantic property corresponding to sentences of the form $\exists \mathbf{x}.\theta(\mathbf{x})$ where θ is *a* r-local formul*a* ?

Example of ongoing work

Local preservation

What is the semantic property corresponding to sentences of the form $\exists \mathbf{x}.\theta(\mathbf{x})$ where θ is a r-local formula ?

Classical model theory

Preservation under *local* elementary embeddings.

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No simple answer yet.

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Classical model theory

Preservation under *local* elementary embeddings.

Finite model theory

No simple answer yet.

Applications to Łoś–Tarski Theorem

“Locally” holds if and only if “globally” holds.

Thank you ☺

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