Do you know what is a preservation theorem?
Preservation Theorems

Stage de M2, MPRI

Aliaume Lopez
11 Juin 2019

Sylvain Schmitz
Jean Goubault-Larrecq
Motivations
Preservation theorem
A monotone formula $\phi \in \text{FO}[\sigma]$ is equivalent to a simple formula $\psi$. 
Preservation theorem
A monotone formula $\phi \in \text{FO}[\sigma]$ is equivalent to a simple formula $\psi$.

Equivalence: Database $\leftrightarrow$ Finite Model
Evaluation of a query on an incomplete database corresponds to evaluation on a family of structures.

1. Existence of a universal model to answer certain answers is equivalent to a preservation theorem (Used in the Chase algorithm (Deutsch et al., 2008)).

2. Naïve evaluation of a query $Q$ yields certain answers if and only if $Q$ is monotone (Gheerbrant et al., 2014).
Motivations

Finite models and logics
A good example is far better than a good precept.

Finite structures over $\sigma \triangleq \{\bullet, \rightarrow, \text{--} \}$

$D \triangleq \{1, 2, 3\}$
$[\bullet] \triangleq \{2\}$
$[\rightarrow] \triangleq \{(1, 2), (3, 3)\}$
$[\text{--}] \triangleq \{(1, 3), (3, 1)\}$

Logical formulas $\text{FO}[\bullet, \rightarrow, \text{--}]$

$\phi := \exists x. \phi \mid \phi \land \phi \mid \neg \phi$
$\mid \bullet x \mid x \rightarrow y$
$\mid x \rightarrow y$
$\exists x. \forall y. \neg((\bullet y) \land \neg(x \rightarrow y))$
The Bible tells us to love our neighbors, and also to love our enemies; probably because generally they are the same people.

**Figure 1 – Locality of FO**
The Bible tells us to love our neighbors, and also to love our enemies; probably because generally they are the same people.

\[ \exists x. \forall y. (x \rightarrow y) \implies (\bullet y) \]

**Figure 1 – Locality of FO**
Chaos is merely order waiting to be deciphered.

Preorders over finite structures

- Induced substructure $\subseteq_i$ Strong Injective Homomorphism
- Substructure $\subseteq$ Injective homomorphism
- Homomorphism $\rightarrow$ Homomorphism
Orders on finite structures

Figure 2 – An investment in knowledge pays the best interest.
$\phi \triangleq \exists x. \deg(x) \geq 3$

Upwards closed

**Figure 3** – Finite graphs encoded using $\Sigma \triangleq \{E\}$
$\phi \triangleq \exists x. \text{deg}(x) \geq 3$

Upwards closed

**Figure 3** – Finite graphs encoded using $\Sigma \triangleq \{E\}$
\[ \phi \triangleq \exists x. \deg(x) \geq 3 \]

Upwards closed

**Figure 3** – Finite graphs encoded using \( \Sigma \triangleq \{E\} \)
\( \phi \triangleq \exists x. \deg(x) \geq 3 \)

**Upwards closed**

**Figure 3** – Finite graphs encoded using \( \Sigma \triangleq \{ E \} \)
\[ \phi \triangleq \exists x. \deg(x) \geq 3 \]

*Upwards closed*

**Figure 3** – Finite graphs encoded using \( \Sigma \triangleq \{E\} \)
Motivations

Preservation theorems
### Known results

<table>
<thead>
<tr>
<th>Ordre</th>
<th>Fragment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\subseteq_i$</td>
<td>EFO</td>
</tr>
<tr>
<td>$\subseteq$</td>
<td>EPFO$\neq$</td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td>EPFO</td>
</tr>
</tbody>
</table>
## Known results

<table>
<thead>
<tr>
<th>( \text{Str}(\sigma) )</th>
<th>Ordre</th>
<th>Fragment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Łós-Tarski ✔</td>
<td>( \subseteq_i )</td>
<td>EFO</td>
</tr>
<tr>
<td>Tarski-Lyndon ✔</td>
<td>( \subseteq )</td>
<td>EPFO(\neq)</td>
</tr>
<tr>
<td>H.P.T. ✔</td>
<td>( \rightarrow )</td>
<td>EPFO</td>
</tr>
</tbody>
</table>
## Known results

<table>
<thead>
<tr>
<th>Str((\sigma))</th>
<th>Ordre</th>
<th>Fragment</th>
<th>FinStr((\sigma))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Łós-Tarski ✓</td>
<td>(\subseteq_i)</td>
<td>EFO</td>
<td>✗ Tait (1959)</td>
</tr>
<tr>
<td>Tarski-Lyndon ✓</td>
<td>(\subseteq)</td>
<td>EPFO(\neq)</td>
<td>✗ Ajtai and Gurevich (1994)</td>
</tr>
<tr>
<td>H.P.T. ✓</td>
<td>(\to)</td>
<td>EPFO</td>
<td>✓ Rossman (2008)</td>
</tr>
</tbody>
</table>
**Lemma**

Preservation theorems \textbf{do not relativise to subclasses.}
Motivations

Classical results
Łós-Tarski’s theorem
Let $\phi$ be a closed formula, preserved under induced substructure. There exists a closed existential formula $\psi$ such that $\phi \iff \psi$. 

Proof

Considérons $T_\forall \equiv \{ \theta \mid \phi \vdash \theta \text{ et } \theta \text{ universelle} \}$. Par construction, $\phi \vdash T_\forall$. Soit $M$ un modèle de $T_\forall$, montrons que $M$ est un modèle de $\phi$. Pour cela, considérons $\{ \phi \} \cup \text{Diag}(M)$. Par l’absurde, cette théorie est incohérente, le théorème de compacité permet d’en extraire une théorie finie incohérente. Or, $\text{Diag}(M)$ est stable par conjonction finie et est cohérente. Ainsi, il existe une formule $\theta \in \text{Diag}(M)$ telle que $\{ \theta, \phi \}$ est incohérente. Par construction, cela veut dire que $\phi \vdash \neg \theta$. Ainsi, $\neg \theta \in T_\forall$, et donc $M \models \neg \theta$, ce qui est absurde. Ainsi, $\{ \phi \} \cup \text{Diag}(M)$ possède un modèle $N$, par construction $M \subseteq N$, $N \models \phi$ donc $M \models \phi$. Par la suite, $\{ \neg \phi \} \cup T_\forall$ est incohérente. Donc en utilisant le théorème de compacité, on déduit que celle-ci possède un sous-ensemble fini incohérent. Comme $T_\forall$ est cohérente, on a donc une formule dans $T_\forall$ qui est équivalente à $\phi$. 

Compactness
Łós-Tarski's theorem
Let $\phi$ be a closed formula, preserved under induced substructure. There exists a closed existential formula $\psi$ such that $\phi \iff \psi$.

Proof

Considérons $T_\forall \triangleq \{ \theta \mid \phi \vdash \theta \text{ et } \theta \text{ universelle } \}$. Par construction, $\phi \vdash T_\forall$. Soit $M$ un modèle de $T_\forall$, montrons qu'il est aussi un modèle de $\phi$. Pour cela, considérons $\{ \phi \} \cup \text{Diag}(M)$. Par absurdité, cette théorie est incohérente, le théorème de la compacité permet d'en extraire une théorie finie incohérente. Or, Diag($M$) est stable par conjonction finie et est cohérente. Ainsi, il existe une formule $\theta \in \text{Diag}(M)$ et $\theta$ et $\phi$ est incohérente.

Par construction, cela veut dire que $\phi \vdash \neg \theta$. Ainsi, $\neg \theta \in T_\forall$, et donc $M \models \neg \theta$, ce qui est absurde.

Ainsi, $\{ \phi \} \cup \text{Diag}(M)$ possède un modèle $N$, par construction $M \subseteq_i N$, $N \models \phi$ donc $M \models \phi$.

Par la suite, $\{ \neg \phi \} \cup T_\forall$ est incohérente. Donc en utilisant le théorème de compacité, on déduit que celle-ci possède un sous-ensemble fini incohérent. Comme $T_\forall$ est cohérente, on a donc une formule dans $T_\forall$ qui est équivalente à $\phi$. 
Involved counter-example

The sad truth
The family $S$ of simple planar graphs using only two labels does not satisfy a preservation theorem for $\subseteq_i$.

Adaptation
Can be (using some tricks) adapted to $\subseteq$. 
Figure 4 – The graph $G_5$
Motivations

Two sides of a same coin.
Lemma
The family $\mathcal{P} = \{P_k \mid k \in \mathbb{N}_{\geq 1}\}$ of finite paths satisfies a preservation theorem for $\subseteq_i$. 


First example: Finite paths

Figure 5 – Evaluation of a monotone formula \( \phi \) over \( \mathcal{P} \)
First example: Finite paths

Figure 5 – Evaluation of a monotone formula $\phi$ over $\mathcal{P}$
First example: Finite paths

**Lemma**
A formula $\phi$ preserved under $\subseteq_i$ on $\mathcal{P}$ is equivalent to

$$\exists x_1, \ldots, \exists x_k. x_1 \neq x_2 \neq \cdots \neq x_k \quad (1)$$

Notes
(i) The order $\subseteq_i$ is total and well founded.
(ii) No property of FO were ever used!
First example: Finite paths

Lemma
A formula $\phi$ preserved under $\subseteq_i$ on $\mathcal{P}$ is equivalent to

$$\exists x_1, \ldots, \exists x_k. x_1 \neq x_2 \neq \cdots \neq x_k \quad (1)$$

Notes
(i) The order $\subseteq_i$ is total and well founded over $\mathcal{P}$
(ii) No property of FO were ever used!
Well Quasi Order / $\text{wqo}$ (e.g. Kruskal, 1972)

**Figure 6** – Every non-empty upwards closed set $U$ has a non-empty finite basis of (finite) minimal elements.
**Well Quasi Order / \(wqo\) (e.g. Kruskal, 1972)**

**Figure 6** – Every non empty upwards closed set \(U\) has a non empty finite basis of (finite) minimal elements.
First example: Generalisation

Well Quasi Order / \( wqo \) (e.g. Kruskal, 1972)

Figure 6 – Every non empty upwards closed set \( U \) has a non empty finite basis of (finite) minimal elements.

Application

\( wqo \implies \text{preservation} \) (2)
Second example: Finite cycles

**Lemma**

The family $\mathcal{C} = \{C_k \mid k \in \mathbb{N}_{\geq 3}\}$ of finite cycles satisfies a preservation theorem for $\subseteq_i$. 
Second example: Finite cycles

Figure 7 – Evaluation of a monotone formula $\phi$ over $C$
Second example: Finite cycles

**Figure 7** – Evaluation of a monotone formula \( \phi \) over \( C \)
Second example: Finite cycles

Figure 7 – Evaluation of a monotone formula $\phi$ over $C$
Second example: Finite cycles

\[ \text{Locality} \]

Figure 7 – Evaluation of a monotone formula \( \phi \) over \( C \)
Lemma
Every formula $\phi$ preserved under $\subseteq_i$ over $\mathcal{C}$ is equivalent to a formula of the following form

\[
\left( \bigvee_{k \in D} \psi_{C_k} \right) \lor \psi_{P_n} \quad (3) \\
\left( \bigvee_{k \in D} \psi_{C_k} \right) \quad (4)
\]

Where $D$ is a finite set of integers below $k$ and $M \models \psi_U \iff U \subseteq_i M$. 

Lemma

Every formula $\phi$ preserved under $\subseteq_i$ over $C$ is equivalent to a formula of the following form

$$(\bigvee_{k \in D} \psi_{C_k}) \lor \psi_{P_n} \quad (3)$$

$$(\bigvee_{k \in D} \psi_{C_k}) \quad (4)$$

Where $D$ is a finite set of integers below $k$ and $M \models \psi_U \iff U \subseteq_i M$.

Notes

(i) The order $\subseteq_i$ is just isomorphism over $C$, which is not wqo.

(ii) Locality of FO is crucial in the construction
Motivations

A landscape complex enough
Figure 2. Inclusion map of some important properties of classes of graphs.

2.4.1. Relational Structures.

A relational vocabulary is a finite set of relation symbols, each with a specified arity. A -structure consists of a universe or domain, and an interpretation which associates to each relation symbol of some arity , a relation .

\[
\begin{align*}
\text{Star forests} & \quad \text{Path forests} \\
\text{Bounded tree-depth} & \quad \text{Forests} \\
\text{Bounded tree-width} & \quad \text{Planar} \\
\text{Bounded genus} & \\
\text{Excluded apex minor} & \\
\text{Excluded minor} & \\
\text{Excluded topological minor} & \\
\text{Uniformly Almost wide} & \\
\text{Bounded expansion} & \\
\text{Bounded local expansion} & \\
\text{Nowhere dense} & \\
\text{Dense} & \\
\end{align*}
\]
Sparsity (Nešetřil and Ossona de Mendez, 2010)

Figure 2. Inclusion map of some important properties of classes of graphs.

2.4.1. Relational Structures.

A relational vocabulary is a finite set of relation symbols, each with a specified arity. A \( A \)-structure consists of a universe or domain, and an interpretation which associates to each relation symbol \( R \) of some arity \( r \), a relation \( R_A \subseteq A^r \).
A structural approach

- Bounded Tree Depth
- Bounded Shrub Depth
- M-partite graph
- Co-graph
- Bounded Tree Width
- Bounded Clique Width
- NLC
- $k$
- $F$
- totally ordered
- wqo
- Brignall et al. (2018)
- Daligault et al. (2010)
Personal Contribution

Logically pre-spectral spaces (LPS)
**logically pre-spectral spaces:**

\[ U = \mathbb{[}\phi\mathbb{]} \]

**Figure 8** – Every non empty and *definable* upwards closed set \( U \) admits a non empty finite basis of minimal (finite) elements.
logically pre-specral spaces: some definitions

Figure 8 – Every non empty and definable upwards closed set $U$ admits a non empty finite basis of minimal (finite) elements.
Link with preservation theorems

(i) If $X$ is a logically pre-spectral space, then $X$ admits a preservation theorem.

(ii) If $X$ admits a preservation theorem and $X$ is downwards closed in $\text{FinStr}(\sigma)$ then $X$ is a logically pre-spectral space.
Looking back on examples

(i) The family $\mathcal{P}$ is a logically pre-spectral space for $\subseteq_i$.

(ii) The family $\mathcal{C}$ is NOT a logically pre-spectral space but admits a preservation theorem.

(iii) The family of graphs of degree bounded by 2 is a logically pre-spectral space for $\subseteq_i$, but is not wqo.
logically pre-specral spaces: some definitions

Figure 9 – A not so well chosen illustration
logically pre-speccral spaces: some definitions

**Figure 9** – A not so well chosen illustration
Personal Contribution

Stability properties (yay!)
Restriction, interpretations

FO-interpretation, surjective, monotone

\[ C \xrightarrow{\Gamma} D \]

Restriction to a subset...

1. To an upwards closed definable set
2. To a downwards closed definable set
### Stability of logically pre-spectral spaces

<table>
<thead>
<tr>
<th>Name</th>
<th>Class</th>
<th>Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disjoint union</td>
<td>$C \cup D$</td>
<td>$A \uplus B$</td>
</tr>
<tr>
<td>Cartesian product</td>
<td>$C \times D$</td>
<td>$A \times B$</td>
</tr>
<tr>
<td>Dot product</td>
<td>$C \cdot D$</td>
<td>$A_1 \uplus \cdots \uplus A_n$</td>
</tr>
<tr>
<td>Finite words</td>
<td>$C^*$</td>
<td>$A_1 \uplus \cdots \uplus A_n$</td>
</tr>
<tr>
<td>Wreath product$^1$</td>
<td>$C \rtimes D$</td>
<td></td>
</tr>
</tbody>
</table>

$^1$ With some restrictions
Personal Contribution

Applications
Figure 10 – Inner of two tables using the equation $\text{NOM} = \text{EMPLOYÉ·E}$
A dense class that is not \textit{wqo}

\textbf{Figure 11} – A (small) element of \((\text{Graph}_{\leq 2})^*\)
A dense class that is not \text{wqo}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{A (small) element of \((\text{Graph}_{\leq 2})^*\)}
\end{figure}
Conclusion
Beautiful results

The ones presented here

1. General framework to derive preservation theorems
2. Stability properties extending known results
3. Caveat : use with care!

« Some battles are silently won »

1. Adaptations of counterexamples to $\subseteq$
2. Adaptations of counterexamples to the canonic $C = \text{Graph}$
3. Study of tree-depth, clique-width, and relationship with wqo.
Some ideas

(i) Query enumeration (Schweikardt et al., 2018)
(ii) Fast formula evaluation (Grohe et al., 2017)
(iii) More powerful logics (Kuske and Schweikardt, 2018)
(iv) Use more topology? (Nešetřil and Ossona de Mendez, 2012, Chapter 10)


