Do you know what is a preservation theorem ?

Preservation Theorems

Stage de M2, MPRI

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Motivations

Preservation theorem
A monotone formula $\phi \in \mathrm{FO}[\sigma]$ is equivalent to a simple formula ψ.

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Equivalence : Database $\leftrightarrow$ Finite Model
Evaluation of a query on an incomplete database corresponds to evaluation on a family of structures.

1. Existence of a universal model to answer certain answers is equivalent to a preservation theorem (Used in the Chase algorithm (Deutsch et al., 2008)).
2. Naïve evaluation of a query Q yields certain answers if and only if Q is monotone (Gheerbrant et al., 2014).

Motivations

Finite models and logics

A good example is far better than a good precept.

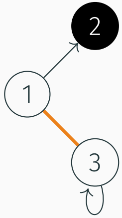
Finite structures over $\sigma \triangleq \{\bullet, \longrightarrow, \text{—}\}$

$$D \triangleq \{1, 2, 3\}$$

$$[\![\bullet]\!] \triangleq \{2\}$$

$$[\![\longrightarrow]\!] \triangleq \{(1,2),(3,3)\}$$

$$[\![\text{—}]\!] \triangleq \{(1,3),(3,1)\}$$



Logical formulas $\mathrm{FO}[\bullet, \longrightarrow, \text{—}]$

$$\begin{aligned}\phi := \exists x.\phi \mid \phi \wedge \phi \mid \neg\phi \\ \mid\ \bullet x \mid x \text{—} y \\ \mid x \longrightarrow y\end{aligned}$$

$$\exists x.\forall y.\neg((\bullet y) \wedge \neg(x \text{—} y))$$

The Bible tells us to love our neighbors, and also to love our enemies; probably because generally they are the same people.

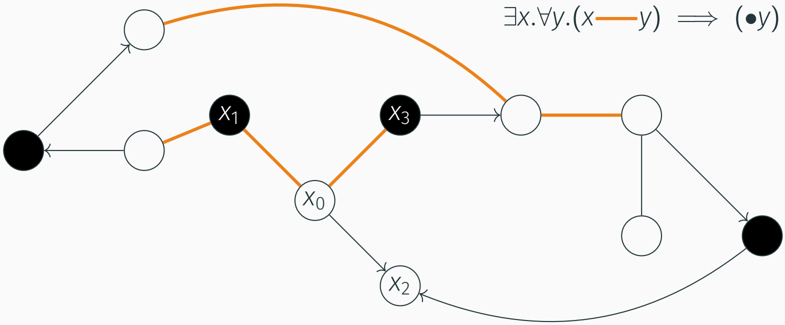


FIGURE 1 – Locality of FO

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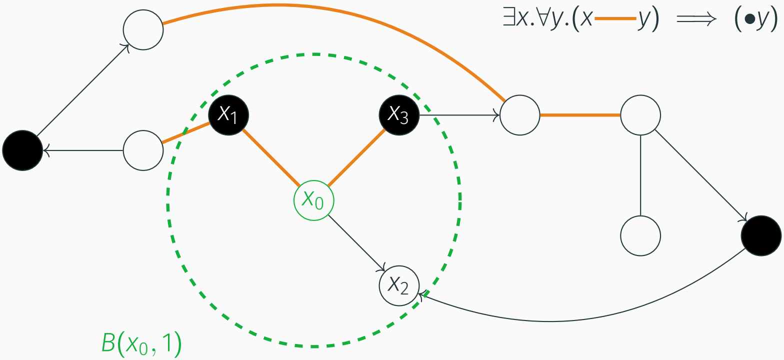


FIGURE 1 – Locality of FO

Preorders over finite structures

Induced substructure	$\subseteq_i$	Strong Injective Homomorphism
Substructure	$\subseteq$	Injective homomorphism
Homomorphism	$\rightarrow$	Homomorphism

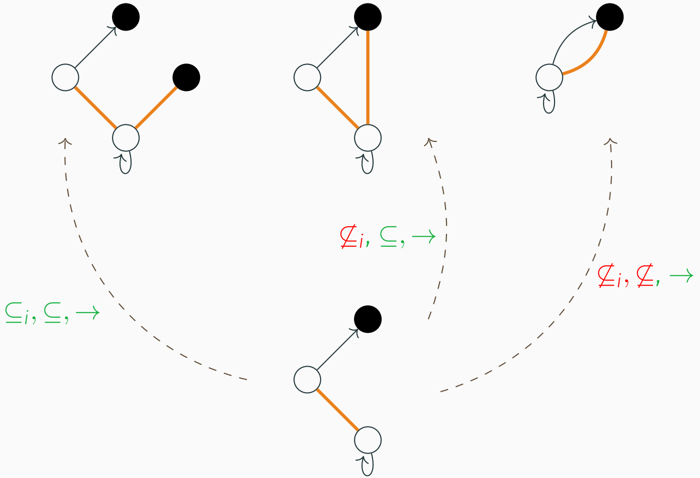


FIGURE 2 – An investment in knowledge pays the best interest.

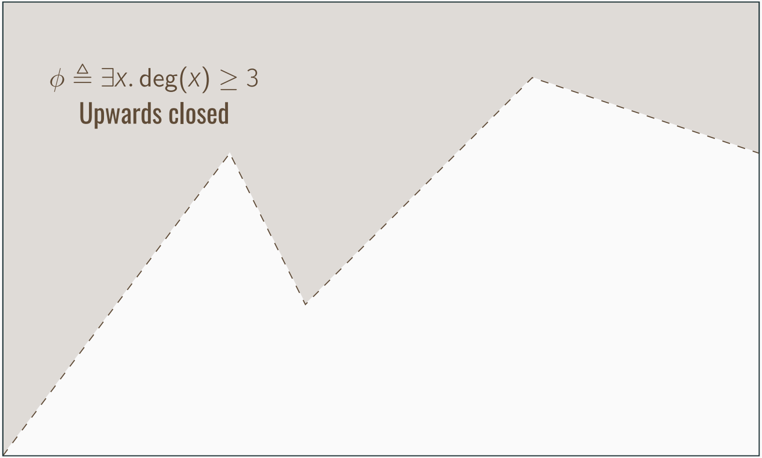
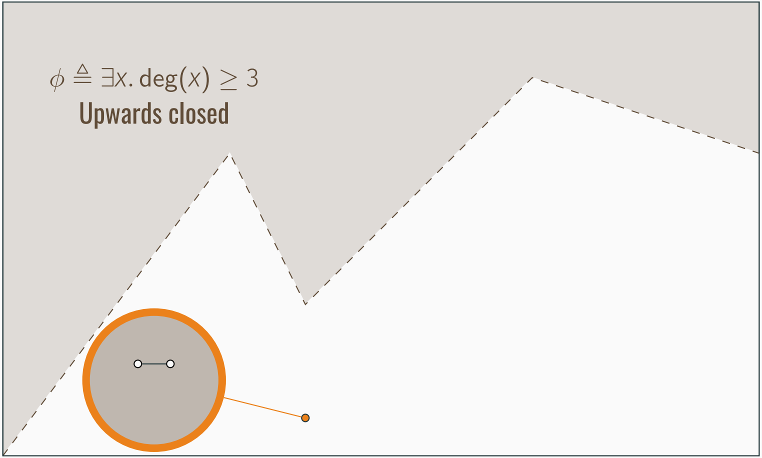


FIGURE 3 – Finite graphs encoded using $\Sigma \triangleq \{E\}$



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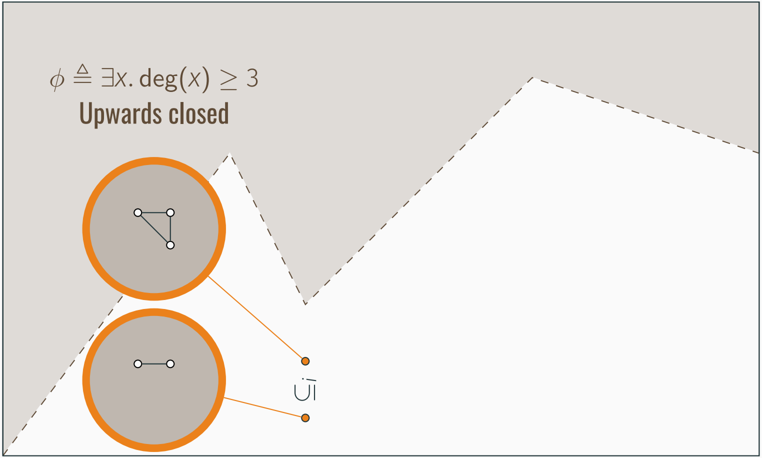

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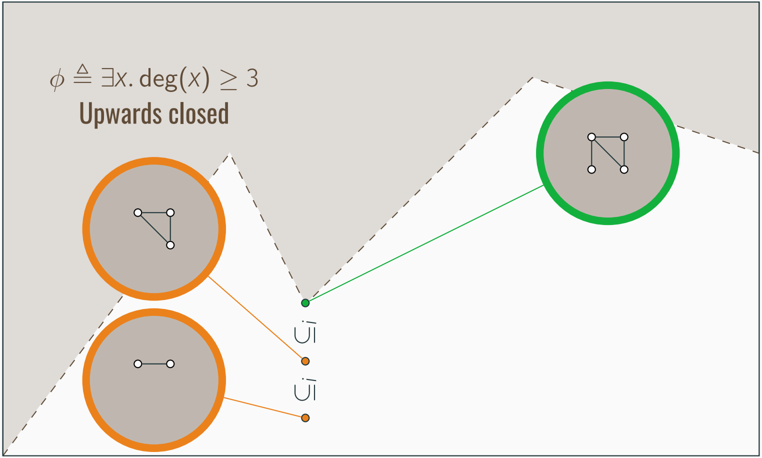

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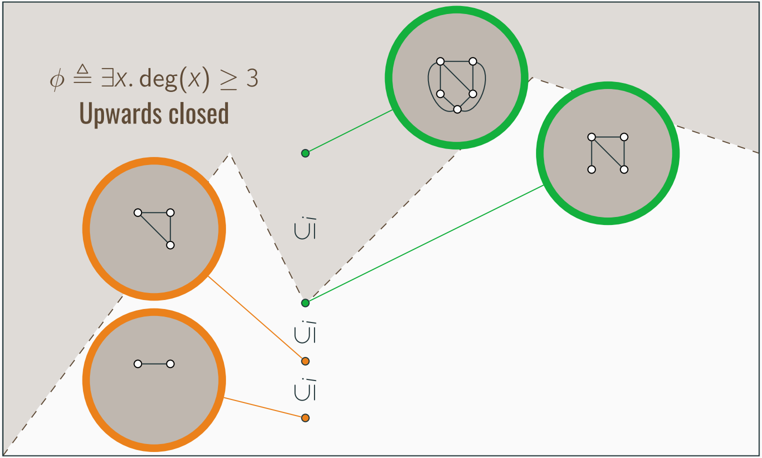

FIGURE 3 – Finite graphs encoded using $\Sigma \triangleq \{E\}$

Motivations

Preservation theorems

Known results

Ordre	Fragment
$\subseteq_i$	EFO
$\subseteq$	$\mathrm{EPFO}^{\neq}$
$\rightarrow$	EPFO

Known results

Str(σ)	Ordre	Fragment
Łós-Tarski ✓	$\subseteq_i$	EFO
Tarski-Lyndon ✓	$\subseteq$	$\mathrm{EPFO}^{\neq}$
H.P.T. ✓	$\to$	EPFO

Link between syntax and semantics

Known results

Str(σ)	Ordre	Fragment	FinStr(σ)
Łós-Tarski ✓	$\subseteq_i$	EFO	✗ Tait (1959)
Tarski-Lyndon ✓	$\subseteq$	EPFO$^{\neq}$	✗ Ajtai and Gurevich (1994)
H.P.T. ✓	$\to$	EPFO	✓ Rossman (2008)




Lemma
Preservation theorems do not relativise to subclasses.

Motivations

Classical results

Łós-Tarski's theorem

Let ϕ be a closed formula, preserved under induced substructure.

There exists a closed existential formula ψ such that $\phi \iff \psi$.

Łós-Tarski's theorem

Let ϕ be a closed formula, preserved under induced substructure. There exists a closed existential formula ψ such that $\phi \iff \psi$.

Proof

Compactness

Considérons $T_\forall \triangleq \{\theta \mid \phi \vdash \theta$ et θ universelle $\}$. Par construction, $\phi \vdash T_\forall$.

Soit M un modèle de $T_\forall$, montrons que M est un modèle de ϕ. Pour cela, considérons $\{\phi\} \cup \mathsf{Diag}(M)$. Par l'absurde, cette théorie est incohérente, le théorème de compacité permet d'en extraire une théorie finie incohérente. Or, $\mathsf{Diag}(M)$ est stable par conjonction finie et est cohérente. Ainsi, il existe une formule $\theta \in \mathsf{Diag}(M)$ telle que $\{\theta, \phi\}$ est incohérente.

Par construction, cela veut dire que $\phi \vdash \neg\theta$. Ainsi, $\neg\theta \in T_\forall$, et donc $M \models \neg\theta$, ce qui est absurde.

Ainsi, $\{\phi\} \cup \mathsf{Diag}(M)$ possède un modèle N, par construction $M \subseteq_i N$, $N \models \phi$ donc $M \models \phi$.

Par la suite, $\{\neg\phi\} \cup T_\forall$ est incohérente. Donc en utilisant le théorème de compacité, on déduit que celle-ci possède un sous ensemble fini incohérent. Comme $T_\forall$ est cohérente, on a donc une formule dans $T_\forall$ qui est équivalente à ϕ.

The sad truth
The family $\mathcal{S}$ of simple planar graphs using only two labels does not satisfy a preservation theorem for $\subseteq_i$.

Adaptation
Can be (using some tricks) adapted to $\subseteq$.

Hide this family that I shall not see

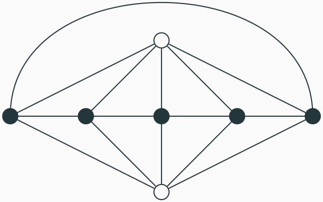

FIGURE 4 – The graph G_5

Motivations

Two sides of a same coin.

Lemma
The family $\mathcal{P} = \{P_k \mid k \in \mathbb{N}_{\geq 1}\}$ of finite paths satisfies a preservation theorem for $\subseteq_i$.

First example : Finite paths

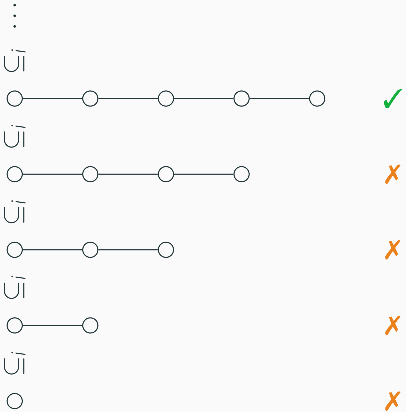

FIGURE 5 – Evaluation of a monotone formula ϕ over $\mathcal{P}$

First example : Finite paths

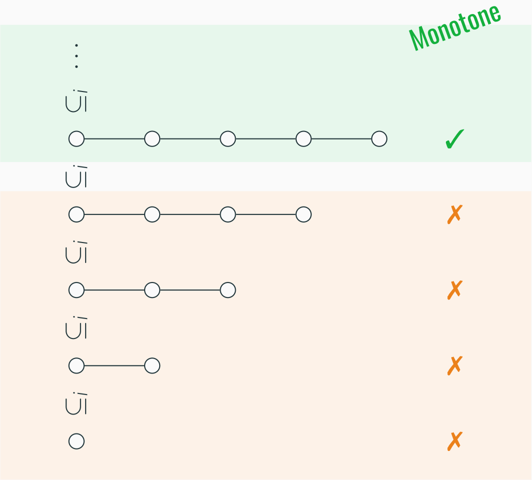

FIGURE 5 – Evaluation of a monotone formula ϕ over $\mathcal{P}$

Lemma
A formula ϕ preserved under $\subseteq_i$ on $\mathcal{P}$ is equivalent to

$$\exists x_1, \dots, \exists x_k . x_1 \neq x_2 \neq \cdots \neq x_k \tag{1}$$

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$$\exists x_1, \ldots, \exists x_k . x_1 \neq x_2 \neq \cdots \neq x_k \quad (1)$$

Notes

(i) The order $\subseteq_i$ is *total* and *well founded* over $\mathcal{P}$

(ii) No property of FO were ever used!

Well Quasi Order / wqo (e.g. Kruskal, 1972)

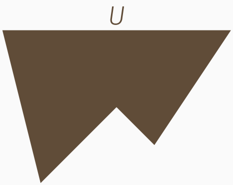
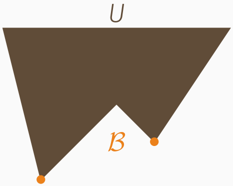


FIGURE 6 – Every non empty upwards closed set U has a non empty finite basis of (finite) minimal elements.

Well Quasi Order / wqo (e.g. Kruskal, 1972)



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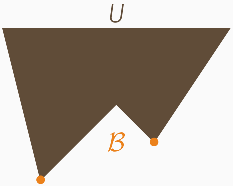

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Application

$$\text{wqo} \implies \text{preservation} \tag{2}$$

Lemma
The family $\mathcal{C} = \{C_k \mid k \in \mathbb{N}_{\geq 3}\}$ of finite cycles satisfies a preservation theorem for $\subseteq_i$.

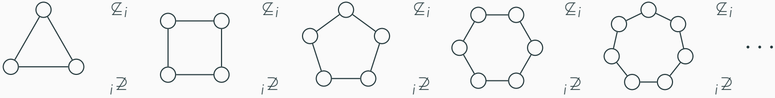
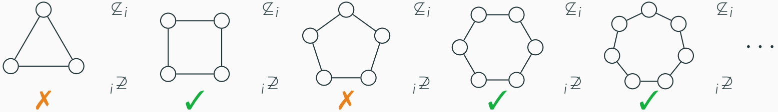


FIGURE 7 – Evaluation of a monotone formula ϕ over $\mathcal{C}$



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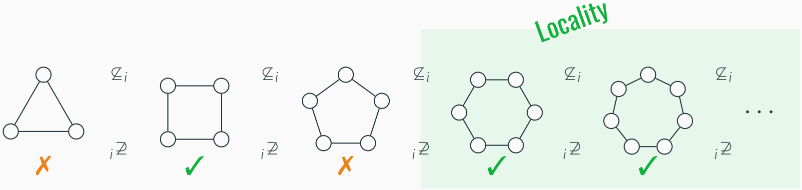

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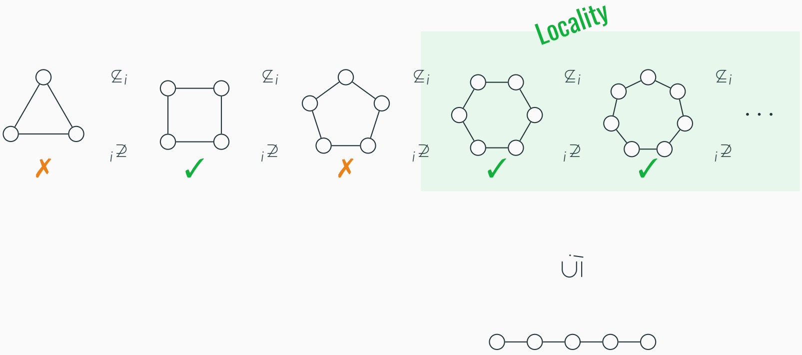

FIGURE 7 – Evaluation of a monotone formula ϕ over $\mathcal{C}$

Lemma

Every formula ϕ preserved under $\subseteq_i$ over $\mathcal{C}$ is equivalent to a formula of the following form

$$\left(\bigvee_{k \in D} \psi_{C_k}\right) \vee \psi_{P_n} \qquad (3) \qquad\qquad \left(\bigvee_{k \in D} \psi_{C_k}\right) \qquad (4)$$

Where D is a finite set of integers below k and $M \models \psi_U \iff U \subseteq_i M$.

Second example : Finite cycles

Lemma

Every formula ϕ preserved under $\subseteq_i$ over $\mathcal{C}$ is equivalent to a formula of the following form

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Notes

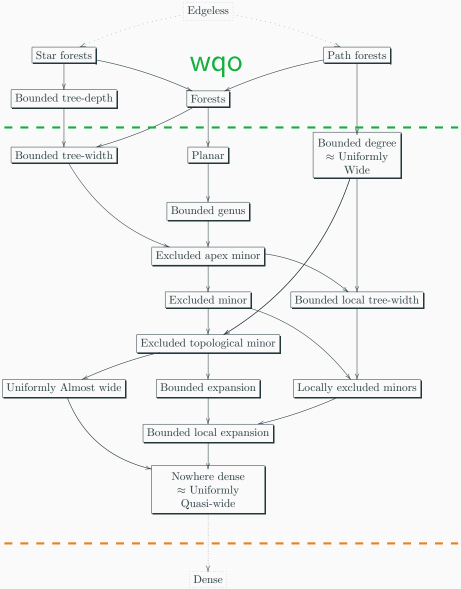
(i) The order $\subseteq_i$ is just isomorphism over $\mathcal{C}$, which is not wqo.

(ii) Locality of FO is crucial in the construction

Motivations

A landscape complex enough

Sparsity (Nešetřil and Ossona de Mendez, 2010)



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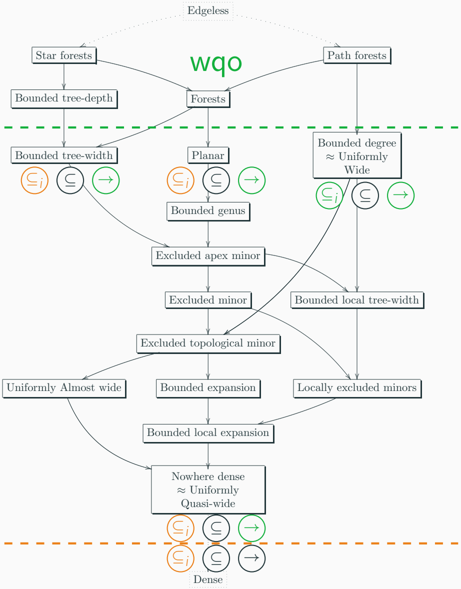
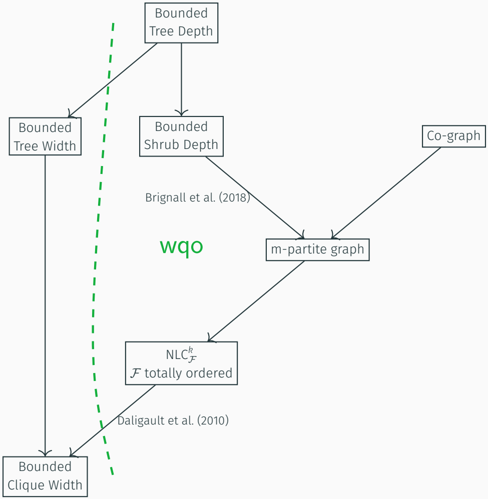


A structural approach



Personal Contribution

Logically pre-spectral spaces (LPS)

logically pre-specral spaces

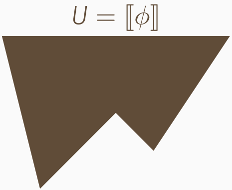


FIGURE 8 – Every non empty and definable upwards closed set U admits a non empty finite basis of minimal (finite) elements.

logically pre-specral spaces

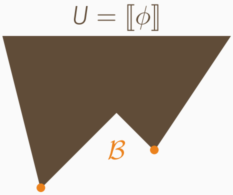


FIGURE 8 – Every non empty and definable upwards closed set U admits a non empty finite basis of minimal (finite) elements.

Link with preservation theorems

(i) If X is a logically pre-spectral space, then X admits a preservation theorems

(ii) If X admits a preservation theorem and X is downwards closed in $\mathsf{FinStr}(\sigma)$ then X is a logically pre-spectral space.

Looking back on examples

(i) The family $\mathcal{P}$ is a logically pre-spectral space for $\subseteq_i$.

(ii) The family $\mathcal{C}$ is NOT a logically pre-spectral space but admits a preservation theorem

(iii) The family of graphs of degree bounded by 2 is a logically pre-spectral space for $\subseteq_i$, but is not wqo.

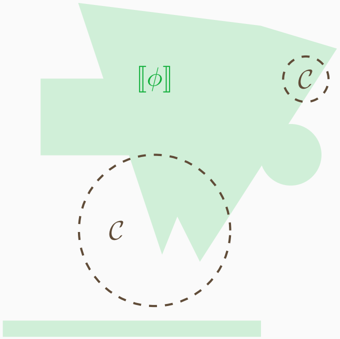


FIGURE 9 – A not so well chosen illustration

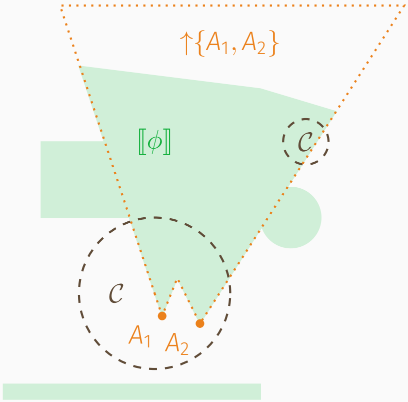


FIGURE 9 – A not so well chosen illustration

Personal Contribution

Stability properties (yay!)

Restriction, interpretations

FO-interpretation, surjective, monotone

$$\mathcal{C} \xrightarrow{\Gamma} \mathcal{D}$$

Restriction to a subset...

1. To an upwards closed definable set
2. To a downwards closed definable set

Algebraic operations!

Stability of logically pre-specral spaces

Name	Class	Elements
Disjoint union	$\mathcal{C} \cup \mathcal{D}$	
Cartesian product	$\mathcal{C} \times \mathcal{D}$	$A \uplus_{<} B$
Dot product	$\mathcal{C} \cdot \mathcal{D}$	$A \times B$
Finite words	$\mathcal{C}^*$	$A_1 \uplus_{<} \cdots \uplus_{<} A_n$
Wreath product [1]	$\mathcal{C} \rtimes \mathcal{D}$	

1. With some restrictions

Personal Contribution

Applications

NOM	ÂGE
Nadine	23
Frank	45
Georges	30
Cécile	29

ENTREPRISE	EMPLOYÉ·E
Facebook	Nadine
EDF	Nadine
Orange	Georges
Google	Cécile

$\bowtie$

NOM	ÂGE	ENTREPRISE
Nadine	23	Facebook
Nadine	23	EDF
Georges	30	Orange
Cécile	29	Google

FIGURE 10 – Inner of two tables using the equation NOM=EMPLOYÉ·E

A dense class that is not wqo

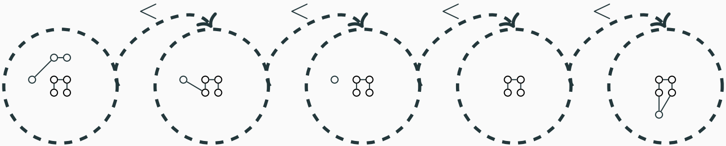


FIGURE 11 – A (small) element of $\left(\mathsf{Graph}_{\leq 2}\right)^*$

A dense class that is not wqo

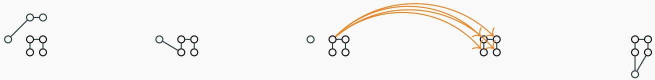


FIGURE 11 – A (small) element of $(\mathsf{Graph}_{\leq 2})^*$

Conclusion

Beautiful results

The ones presented here

1. General framework to derive preservation theorems
2. Stability properties extending known results
3. Caveat : use with care !

« Some battles are silently won »

1. Adaptations of counterexamples to $\subseteq$
2. Adaptations of counterexamples to the canonic $\mathcal{C} = \mathsf{Graph}$
3. Study of tree-depth, clique-width, and relationship with wqo.

What next ?

Some ideas

(i) Query enumeration (Schweikardt et al., 2018)

(ii) Fast formula evaluation (Grohe et al., 2017)

(iii) More powerful logics (Kuske and Schweikardt, 2018)

(iv) Use more topology ? (Nešetřil and Ossona de Mendez, 2012, Chapter 10)

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