# $\mathbb{Z}$-POLYREGULAR FUNCTIONS

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Tuesday, June 27, 2023

LICS 2023, Boston, Massachusetts, USA

# Regularity and aperiodicity for languages

Finite Automaton

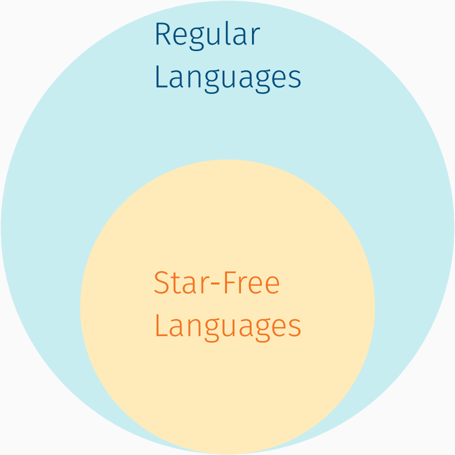
Finite Monoid

MSO Sentence

Counter-free Automaton

Aperiodic Monoid

FO Sentence



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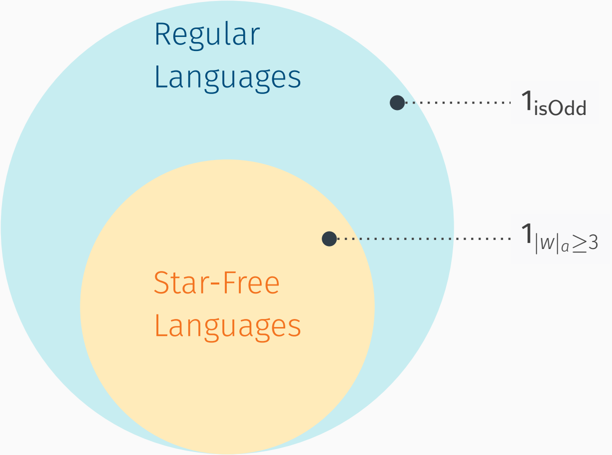

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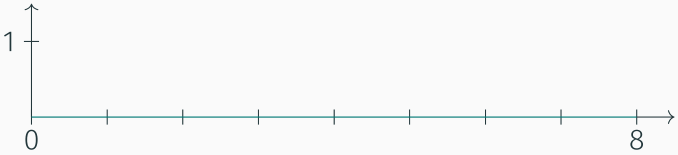


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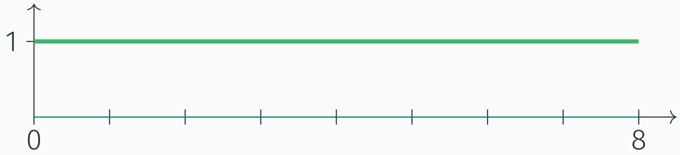


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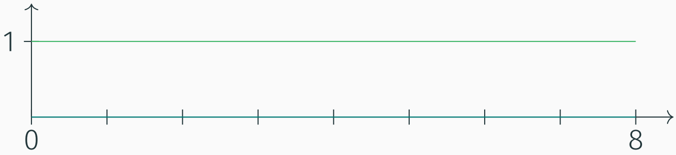


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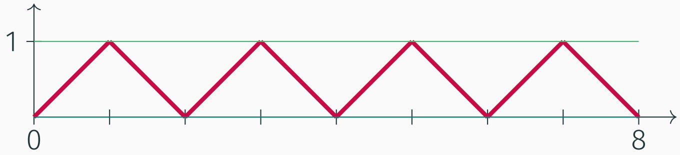


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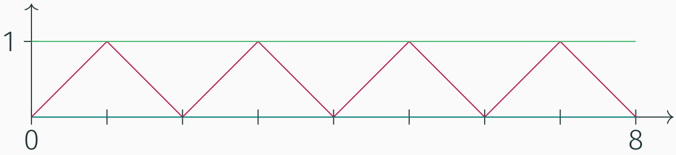


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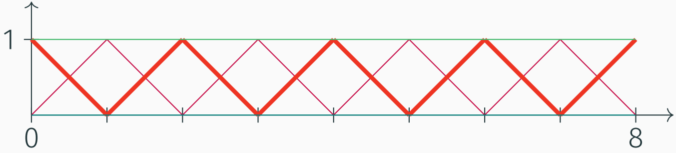


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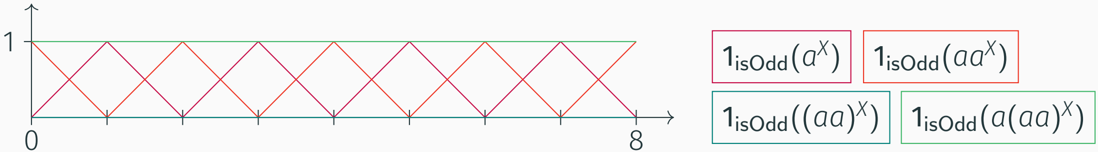

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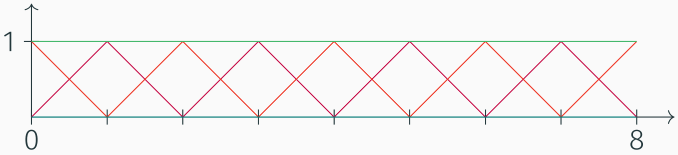


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$\mathbb{1}_{|w|_a \geq 3}$

**Star-free is Decidable!**

Because one can compute the DFA/the syntactic monoid: canonical model.

# WHAT ABOUT FUNCTIONS?

**Regularity for word-to-number functions**

$$\begin{aligned}\text{weighted automata} &\iff \text{rational series}\\ &\iff \text{weighted logics}\\ &\iff \text{rational expressions}\end{aligned}$$

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**Not like previous attempts!**

Both [Reutenauer, 1980, III.4.4] and [Droste and Gastin, 2019] capture indicator functions of non-aperiodic languages.

## Generalisation

$f\colon \Sigma^* \to \mathbb{Z}$ is star-free when

$$\forall u, v, w \in \Sigma^*. \quad \exists P_{u,v,w}(X) \in \mathbb{Z}[X]. \quad f(uw^X v) \simeq P_{u,v,w}(X)$$

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**For rational series, this implies polynomial growth!**

$$\exists k \in \mathbb{N}. \quad \sup_{|w|=n} |f(w)| = \mathcal{O}(n^k) \quad .$$

# This talk:

$\mathbb{Z}$-rational series: $f\colon \Sigma^\star \to \mathbb{Z}$

With polynomial growth: $\exists k \in \mathbb{N}, |f(w)| = \mathcal{O}(|w|^k)$.

A lot of effective translations.

Decidable notion of star-free!

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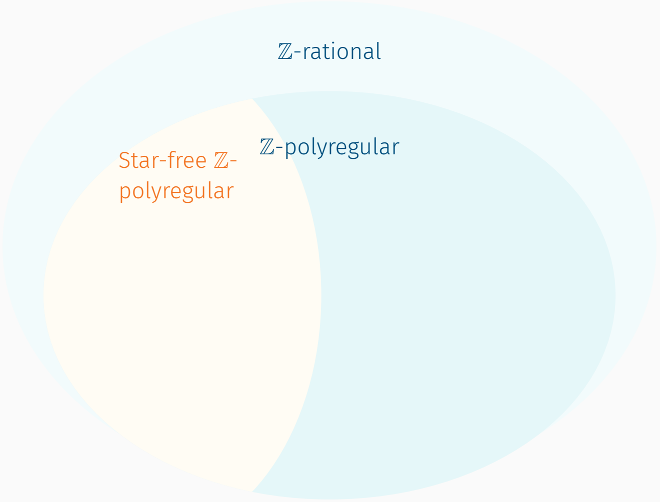

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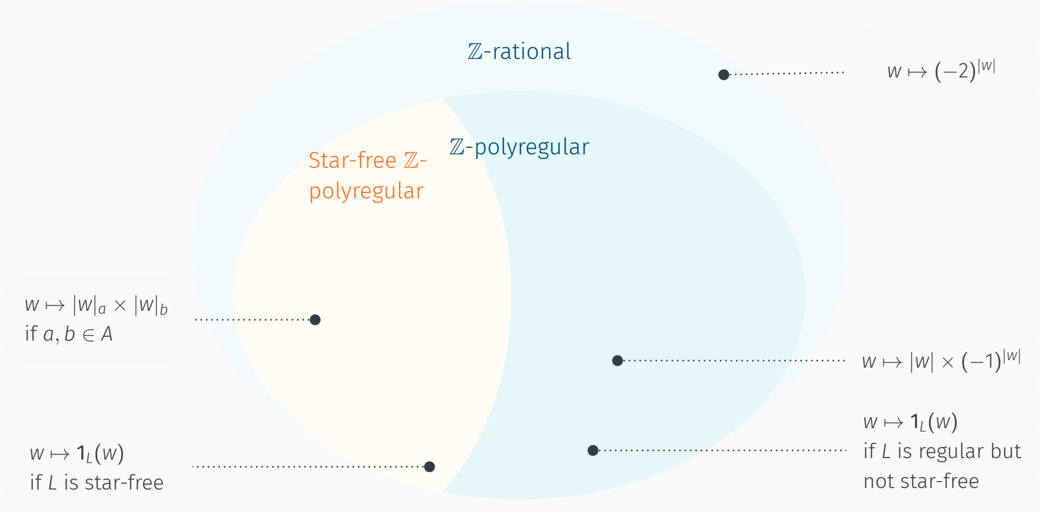

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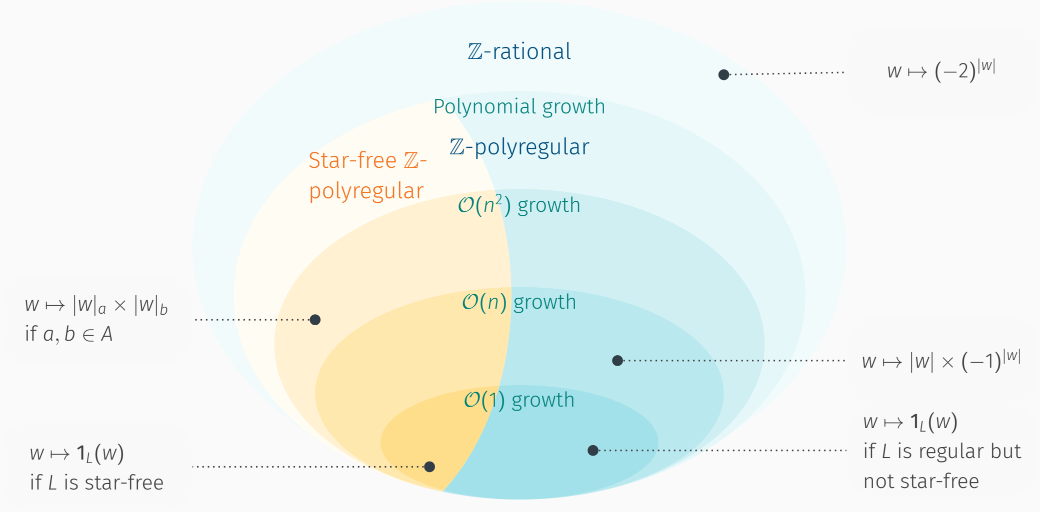

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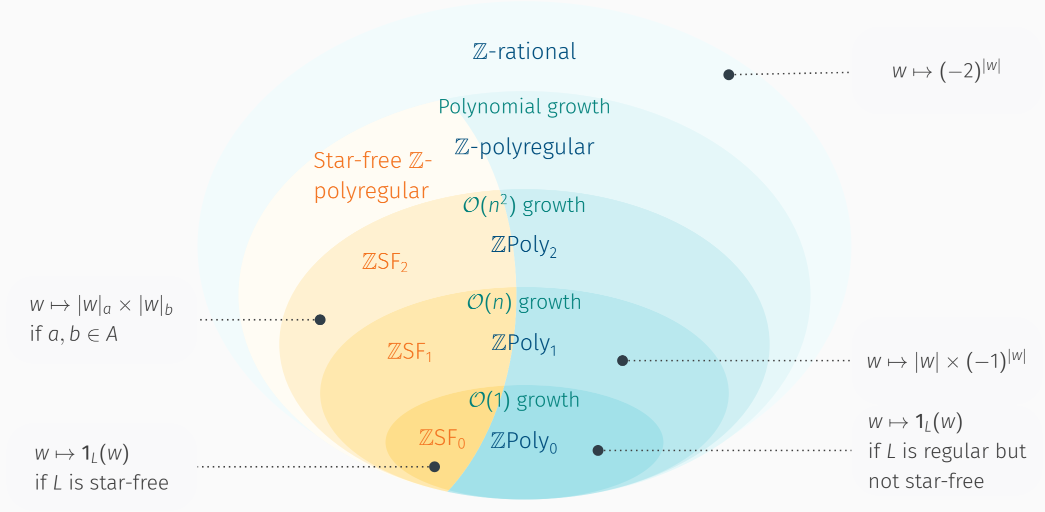

# Our Results: (effectively) equivalent descriptions of the classes

**$\mathbb{Z}$-polyregular functions**

- Counting MSO formulas
- Weighted automata with eigenvalues 0 or roots of unity ($\simeq$ non-trivial groups)
- Closure of regular languages under classical operators
- *Deterministic suffix transducers*

**Star-free $\mathbb{Z}$-polyregular functions**

- Counting FO formulas
- Weighted automata with eigenvalues 0 or 1 ($\simeq$ no non-trivial groups)
- Closure of star-free languages under classical operators
- Counter-free *deterministic suffix transducers*

# A Concrete Deterministic Suffix Transducer

Non star-free function: $f(w) \stackrel{def}{=} (-1)^{|w|} \times |w|$.

Residuals of $f$ up to constant growth rate

Residual transducer of the function.

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$$f(aw) - f(w) = (-1)^{|w|+1} \times (1 + 2|w|) \quad \text{linear growth}$$
$$f(aaw) - f(w) = 2 \times (-1)^{|w|} \quad \text{constant growth}$$

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Residual transducer of the function.

- Deterministic machine whose states corresponds to the *residuals* of $f$
- Output labels are functions of smaller growth applied on suffixes

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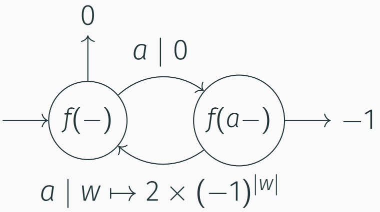

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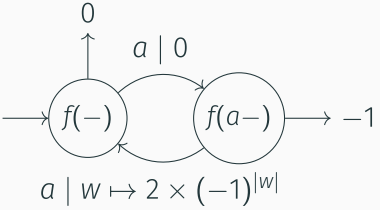


*aaaaa*

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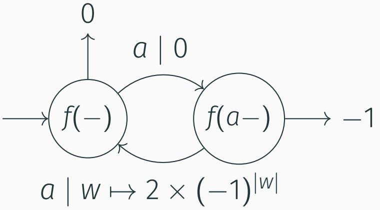


Not *counter-free*: there is a counter between $f(-)$ and $f(a-)$.

**Decide if $\mathbb{N}$-polyregular is star-free $\mathbb{N}$-polyregular?**

The canonical deterministic suffix transducer *cannot* be build in $\mathbb{N}$.

**Star-free rational series?**

Generalize the notions to functions with exponential growth.

# Thank You!

Droste, M. and Gastin, P. (2019).
**Aperiodic weighted automata and weighted first-order logic.**
In *44th International Symposium on Mathematical Foundations of Computer Science, MFCS 2019*, volume 138.

Reutenauer, C. (1980).
**Séries formelles et algèbres syntactiques.**
*Journal of Algebra*, 66(2):448–483.