Fixed Points and Noetherian Topologies

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Abstract

Noetherian spaces are a generalisation of well-quasi-ordering to topologies, that can be used to prove termination of programs. They find applications in the verification of transitions systems, that are better described using topology. The goal of this paper is to allow the systematic description of computations using inductively defined datatypes using Noetherian spaces. This is achieved through a fixed point theorem based on a topological minimal bad sequence argument.

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1 Introduction

The goal of this paper is to bring inductively defined datatypes to the theory of Noetherian spaces. Before that, let us give some context on the history and relevance of this concept.

Well-quasi-orderings. Let $(\mathcal{E}, \leq)$ be a set endowed with a quasi-order. A sequence $(x_n)_n \in \mathcal{E}^{\mathbb{N}}$ is *good* whenever there exists $i < j$ such that $x_i \leq x_j$. A quasi-ordered set $(\mathcal{E}, \leq)$ is a *well-quasi-ordered* if every sequence is good. By calling a sequence bad whenever it is not good, well-quasi-orderings are equivalently defined as having no infinite bad sequences. This generalisation of well-founded total orderings can be used as a basis for proving program termination. For instance, algorithms alike Example 1.1 can be studied via well-quasi-ordering and the length of their bad sequences [5]. More generally, one can map the states of a run to a wqo via a so-called quasi-ranking function to both prove the termination of the program and gain information about its runtime [26, Chapter 2].

▶ **Example 1.1.** Let Alg be the algorithm with three integer variables a, b, c that does one of the following operations $l\colon \langle a, b, c\rangle \leftarrow \langle a-1, b, 2c\rangle$; $r\colon \langle a, b, c\rangle \leftarrow \langle 2c, b-1, 1\rangle$; until a, b or c becomes negative.

For every choice of $a, b, c \in \mathbb{N}^3$, the algorithm Alg builds a bad sequence of triples when ordering $\mathbb{N}^3$ with $(a_1, b_1, c_1) \leq (a_2, b_2, c_2)$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, and $c_1 \leq c_2$. Because $(\mathbb{N}^3, \leq)$ is a well-quasi-ordering [see Dickson's Lemma in 27], Alg terminates for every choice of initial triple $(a, b, c) \in \mathbb{N}^3$.

As a combinatorial tool, well-quasi-orderings appear frequently in varying fields of computer science, ranging from graph theory to number theory [17, 21, 20, 3]. Well-quasi-orderings have also been highly successful in proving the termination of verification algorithms. One critical application of well-quasi-orderings is to the verification of infinite state transition systems, via the study of so-called Well-Structured Transition Systems (WSTS) [2, 1, 14, 7].

Noetherian spaces. One major roadblock arises when using well-quasi-orders: the powerset of a well-quasi-order may fail to be one itself [25]. This is particularly problematic in the study of WSTS, where the powerset construction appears frequently [18, 28, 2]. To tackle this issue, one can justify that the quasi-orders of interest are not pathological, and are actually better quasi-orders [24, 22]. Another approach is offered by the topological notion of *Noetherian space*, which as pointed out by Goubault-Larrecq, can act as a suitable generalisation of well-quasi-orderings that is preserved under the powerset construction [10].

The topological analogues to WSTS enjoy similar decidability properties, and there even exists an analogue to Karp and Miller's forward analysis for Petri nets [11]. Moreover, their topological nature allows to verify systems beyond the reach of quasi-orderings, such as lossy concurrent polynomial programs [11]. This is possible because the polynomials are handled via results from algebraic geometry, through the notion of the *Zariski topology* over $\mathbb{C}^n$ [12, Exercise 9.7.53].

One drawback of the topological approach is that many topologies correspond to a single quasi-ordering. Hence, when the problem is better described via an ordering, one has to choose a specific topology, and there usually does not exist a finest one that is Noetherian.

Inductively defined datatypes. As for well-quasi-orders, Noetherian spaces are stable under finite products and finite sums [27, 12]. While this can be enough to describe the set of configurations of a Petri net using $\mathbb{N}^k$, it does not allow to talk about more complex data structures such as channels, lists, or trees.

In the realm of well-quasi-orderings, the specific cases of finite words and finite trees are handled respectively via Higman's Lemma [17] and Kruskal's Tree Theorem [21]. Let us recall that a word u *embeds* into a word w (written $u \leq_* v$) whenever whenever there exists a strictly increasing map $h\colon |w| \to |w'|$ such that $w_i \leq w_{h(i)}$ for $1 \leq i \leq |w|$. Similarly, a tree t *embeds* into a tree t' (written $t \leq_{\mathsf{tree}} t'$) whenever there exists a map from nodes of t to nodes of t' respecting the least common ancestor relation, and increasing the colours of the nodes. Proofs that finite words and finite trees preserve well-quasi-orderings typically rely on a so-called *minimal bad sequence argument* due to Nash-Williams [23]. However, the argument is quite subtle, and needs to be handled with care [9, 29]. In addition, the argument is not compositional and has to be slightly modified whenever a new inductive construction is desired [e.g., 4, 3].

This picture has been adapted to the topological setting by proposing analogues of the word embedding and tree embedding, together with a proof that they preserve Noetherian spaces [12, Section 9.7]. However, both the definitions and the proofs have an increased complexity, as they rely on an adapted "topological minimal bad sequence argument" that appears to be even more subtle.

One could expect the situation to be more regular. In an ML-like language, one can define words over an alphabet of type `'a` via a type declaration of the form `'a word = Nil | Cons of 'a * 'a word`, and trees over an alphabet of type `'a` via `'a tree = Node of 'a * ('a tree list)`. In a more set-theoretical mindset, one would write `Nil` as the singleton set $\mathbf{1} := \{\star\}$, $A + B$ the disjoint union of A and B, and $A \times B$ their product. An inductive type would then be defined via a least fixed point operator: $\mathsf{lfp}_X.F(X)$. In this language, $\Sigma^* \equiv \mathsf{lfp}_X.\mathbf{1} + \Sigma \times X$, and $\mathsf{T}(\Sigma) \equiv \mathsf{lfp}_X.\Sigma \times X^*$. In the case of well-quasi-orderings, two generic fixed point constructions have already been proposed [16, 8]. In these frameworks, the constructor F in $\mathsf{lfp}_X.F(X)$ has to be a "well-behaved" functor of quasi-orders in order for $\mathsf{lfp}_X.F(X)$ to be a well-quasi-order. Both proposals, while relying on different categorical notions, successfully recover Higman's word embedding and Kruskal's tree embedding via the least fixed point definitions of words and trees. As a side effect, they reinforce the idea that the two quasi-orders are somehow canonical.

In the case of Noetherian spaces, no equivalent framework exists to build inductive datatypes, and the notions of "well-behaved" constructors from [16, 8] rule out the use of important Noetherian spaces, as they require that an element $a \in F(X)$ has been built using *finitely many* elements of X: while this is the case for finite words and finite trees, it does not hold for instance for the arbitrary powerset. Moreover, there have been recent advances in placing Noetherian topologies over spaces that are not straightforwardly obtained through "well-behaved" definitions, such as infinite words [13], or even ordinal length words [15].

1.1 Contributions of this paper

In this paper, we propose a least fixed point theorem for Noetherian topologies. The main contribution of this paper is to build topologies defined inductively over a set X. This is done in a way that greatly differs from the categorical frameworks proposed in the study of well-quasi-orders [16, 8], as the construction of the space is entirely *decoupled* from the construction of the topology. In particular, the set X itself need not be inductively defined.

In this setting, we consider a fixed set X and a map R from topologies τ over X to topologies $R(\tau)$ over X. Because the set of topologies over X is a complete lattice, it suffices to ask for R to be monotone to guarantee that it has a least fixed point, that we write $\mathsf{lfp}_{\tau}.R(\tau)$. In general, this least fixed point will not be Noetherian, but we show that a simple sufficient condition on F guarantees that it is. This main theorem (Theorem 3.21), encapsulates all the complexity of the topological adaptations of the minimal bad sequences arguments [12, Section 9.7], and we believe that it has its own interest.

The necessity to separate the construction of the set of points from the construction of the topology might be perceived as a weakness of the theory, when it is in fact a strength of our approach. We illustrate this by giving a shorter proof that the words of ordinal length are Noetherian [15], without providing an inductive definition of the space. As an illustration of the versatility of our framework, we introduce a reasonable topology over ordinal branching trees (with finite depth), and prove that it is Noetherian using the same technique.

In the specific cases where the space of interest can be obtained as a least fixed point of a "well-behaved" functor, we show how Theorem 3.21 can be used to generalise the categorical framework of Hasegawa [16] to a topological setting.

Outline. In Section 2 we recall some of the main results in the theory of Noetherian spaces. In Section 3 we prove our main result (Theorem 3.21). In Section 4 we explore how this result covers existing topological results in the literature, and provide a new non-trivial Noetherian space (Definition 4.14). In Section 5, we leverage our main result to devise a Noetherian topology over inductively defined datatypes (Theorem 5.10), and prove that this generalises the work of Hasegawa over well-quasi-orders (Theorem 5.23).

2 A Quick Primer on Noetherian Topologies

A *topological space* is a pair $(\mathcal{X}, \tau)$ where $\tau \subseteq \mathbb{P}(X)$, τ is stable under finite intersections, and τ is stable under arbitrary unions. Before formally introducing the topological counterpart to well-quasi-orderings, let us provide a small dictionary from topology to orders. Given a quasi-ordered set $(\mathcal{E}, \leq)$, a set U is *upwards-closed* whenever $x \in U$ and $x \leq y$ implies $y \in U$ for every $x, y \in X$.

Constructor	**Syntax**	**Topology**
Well-quasi-orders	$\mathcal{E}$	Alexandroff topology
Complex vectors	$\mathbb{C}^k$	Zariski topology
Disjoint sum	$\mathcal{X}_1 + \mathcal{X}_2$	co-product topology
Product	$\mathcal{X}_1 \times \mathcal{X}_2$	product topology
Finite words	$\mathcal{X}^*$	subword topology
Finite trees	$\mathsf{T}(\mathcal{X})$	tree topology
Finite multisets	$\mathcal{X}^{\circledast}$	multiset topology
Transfinite words	$\mathcal{X}^{<\alpha}$	transfinite subword topology
Powerset	$\mathsf{P}(\mathcal{X})$	Lower-Vietoris

Table 1 An algebra of Noetherian spaces.

▶ **Definition 2.1.** *Let $(\mathcal{E}, \leq)$ be a quasi-order. The* Alexandroff topology $\mathsf{alex}(\leq)$ *over $\mathcal{E}$ is the collection of upwards-closed subsets of $\mathcal{E}$.*

▶ **Definition 2.2.** *Let $(\mathcal{X}, \tau)$ be a topological space. The* specialisation preorder $\leq_\tau$ *is defined via $x \leq_\tau y$ whenever for every open $U \in \tau$, if $x \in U$ then $y \in U$.*

It is an easy check that the specialisation pre-order of the Alexandroff topology of a quasi-order $\leq$ is the quasi-order itself. This allows to build intuition by getting back and forth between topologies and quasi-orders. Several topologies can share the same specialisation pre-order $\leq$, among those, the Alexandroff topology is the finest.

We can now build the topological analogue to wqos through the notion of compactness: a subset K of $(\mathcal{X}, \tau)$ is defined as *compact* whenever from every family $(U_i)_{i \in I}$ of open sets such that $K \subseteq \bigcup_{i \in I} U_i$, one can extract a finite subset $J \subseteq_f I$ such that $K \subseteq \bigcup_{i \in J} U_i$. A quasi-order $(\mathcal{E}, \leq)$ is wqo if and only if every subset K of $\mathcal{E}$ is compact. Generalising this property to arbitrary topological spaces $(\mathcal{X}, \tau)$, a topological space $(\mathcal{X}, \tau)$ is said to be a *Noetherian space* whenever every subset of $\mathcal{X}$ is compact.

▶ Remark 2.3. A space (X, τ) is Noetherian if and only if for every increasing sequence of open subsets $(U_i)_{i \in \mathbb{N}}$, there exists $j \in \mathbb{N}$ such that $\bigcup_{i \in \mathbb{N}} U_i = \bigcup_{i \leq j} U_i$.

Following the ideas of wqos, an algebra of Noetherian spaces has been developed and is described in Table 1 [see 10, 12, 15].

3 Refinements of Noetherian topologies

Let us fix a set X equipped the *trivial topology* $\tau_{\text{triv}} := \{\emptyset, X\}$. This space is Noetherian because there are finitely many open sets. The approach taken in this paper is to iteratively refine this topology while keeping it Noetherian, and ultimately prove that the *limit* of this construction remains Noetherian.

▶ **Definition 3.1.** *A* refinement function *over a set X is a function R mapping topologies over X to topologies over X. Moreover, we assume that $\mathsf{R}(\tau)$ is Noetherian whenever τ is, and that $\mathsf{R}(\tau) \subseteq \mathsf{R}(\tau')$ when $\tau \subseteq \tau'$.*

The collection of topologies over a set X is itself a set, and forms a complete lattice for inclusion. Thanks to Tarski's fixed point theorem, every refinement function R has a least fixed point, which can be obtained by transfinitely iterating R from the trivial topology. Let us write $\mathsf{lfp}_\tau.\mathsf{R}(\tau)$ for the least fixed point of R.

Given a quasi-order $(\mathcal{E}, \leq)$ and a set $E \subseteq \mathcal{E}$, let us define the *upwards-closure* of E, written $\uparrow_\leq E$, as the set of elements that are greater or equal than some element of E in $\mathcal{E}$.

▶ **Example 3.2** (Natural Numbers)**.** Over $X := \mathbb{N}$, one can define $\mathsf{Div}(\tau)$ as the topology generated by the sets $\uparrow_\leq (U+1)$ for $U \in \tau$. Then $\mathsf{Div}(\tau_{\mathrm{triv}}) = \{\emptyset, \uparrow_\leq 1, \mathbb{N}\}$, $\mathsf{Div}^2(\tau_{\mathrm{triv}}) = \{\emptyset, \uparrow_\leq 1, \uparrow_\leq 2, \mathbb{N}\}$. More generally, for every $k \geq 0$, $\mathsf{Div}^k(\tau_{\mathrm{triv}}) = \{\emptyset, \uparrow_\leq 1, \ldots, \uparrow_\leq k, \mathbb{N}\}$. It is an easy check that $\mathsf{lfp}_\tau.\mathsf{Div}(\tau)$ is precisely $\mathsf{alex}(\leq)$, which is Noetherian because $(\mathbb{N}, \leq)$ is a well-quasi-ordering.

Not all refinement functions behave as nicely as in Example 3.2, and one can obtain non-Noetherian topologies via their least fixed points.

3.1 An ill-behaved refinement function

Let us consider for this section $\Sigma := \{a, b\}$ with the discrete topology, i.e., $\{\emptyset, \{a\}, \{b\}, \Sigma\}$. Let us now build the set Σ^* of finite words over Σ. Whenever U and V are subsets of Σ^*, let us write UV for their concatenation, defined as $\{uv \colon u \in U, v \in V\}$.

▶ **Definition 3.3.** *Let $\mathsf{R}_{\mathrm{pref}}$, be the function mapping a topology τ over Σ^* to the topology generated by the sets UV where $U \subseteq \Sigma$ and $V \in \tau$,*

We refer to Figure 1 for a graphical presentation of the first two iterations of the refinement function $\mathsf{R}_{\mathrm{pref}}$. For the sake of completeness, let us compute $\mathsf{lfp}_\tau.\mathsf{R}_{\mathrm{pref}}(\tau)$, which is the Alexandroff topology of the prefix ordering on words. Beware that this is not the usual notion of "prefix topology" in the literature [see 6, 12, resp. Section 8 and Exercice 9.7.36].

▶ **Definition 3.4.** *The* prefix topology *τ_{pref^*}, over Σ^* is generated by the following open sets: $U_1 \ldots U_n \Sigma^*$, where $n \geq 0$ and $U_i \subseteq \Sigma$.*

▶ **Lemma 3.5.** *The prefix topology over Σ^* is the least fixed point of $\mathsf{R}_{\mathrm{pref}}$.*

Proof. Consider a subbasic open set $W \in \mathsf{R}_{\mathrm{pref}}(\tau_{\mathrm{pref}^*})$. It is of the form UV with $U \subseteq \Sigma$ and $V \in \tau_{\mathrm{pref}^*}$. Hence, $UV \in \tau_{\mathrm{pref}^*}$. We have proven that, $\mathsf{R}_{\mathrm{pref}}(\tau_{\mathrm{pref}^*}) \subseteq \tau_{\mathrm{pref}^*}$.

Conversely, consider a subbasic open set $W \in \tau_{\mathrm{pref}^*}$. Either it is $\emptyset$, or Σ^*, in which case it trivially belongs to $\mathsf{lfp}_\tau.\mathsf{R}_{\mathrm{pref}}(\tau)$. Or it is of the form $U_1 \ldots U_n \Sigma^*$, with $U_i \subseteq \Sigma$ for $1 \leq i \leq n$, in which case one proves by induction over n that it belongs to ${\mathsf{R}_{\mathrm{pref}}}^n(\tau_{\mathrm{triv}})$. ◀

▶ **Lemma 3.6.** *The function $\mathsf{R}_{\mathrm{pref}}$ is a refinement function.*

Proof. It is an easy check that whenever $\tau \subseteq \tau'$, $\mathsf{R}_{\mathrm{pref}}(\tau) \subseteq \mathsf{R}_{\mathrm{pref}}(\tau')$. Now, assume that τ is Noetherian, it remains to prove that $\mathsf{R}_{\mathrm{pref}}(\tau)$ remains Noetherian. Consider a subset $E \subseteq \Sigma^*$ and let us prove that E is compact in $\mathsf{R}_{\mathrm{pref}}(\tau)$.

For that, we consider an open cover $E \subseteq \bigcup_{i \in I} W_i$, where $W_i \in \mathsf{R}_{\mathrm{pref}}(\tau)$. Thanks to Alexander's subbase lemma, we can assume without loss of generality that W_i is a subbasic open set of $\mathsf{R}_{\mathrm{pref}}(\tau)$, that is, $W_i = U_i V_i$ with $U_i \subseteq \Sigma$ and $V_i \in \tau$.

Since $(\Sigma^*, \tau) \times (\Sigma^*, \tau)$ is Noetherian (see Table 1), there exists a finite set $J \subseteq I$ such that $\bigcup_{i \in J} U_i \times V_i = \bigcup_{i \in I} U_i \times V_i$. This implies that $E \subseteq \bigcup_{i \in J} U_i V_i$, and provides a finite subcover of E. ◀

The sequence $\bigcup_{0 \leq i \leq k} a^i b \Sigma^*$, for $k \in \mathbb{N}$, is a strictly increasing sequence of opens. Therefore, the prefix topology is not Noetherian. The terms $a^i b \Sigma^*$ can be observed in Figure 1 as a diagonal of incomparable open sets.

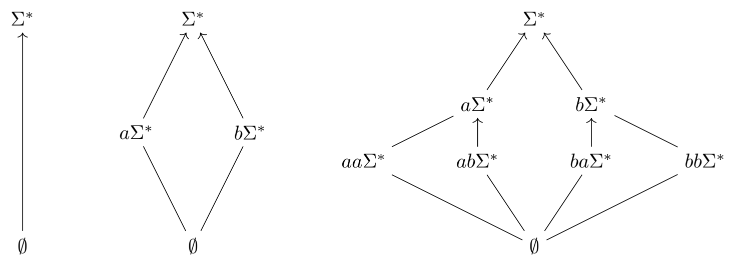

Figure 1 Iterating R_{pref} over Σ^*. On the left the trivial topology τ_{triv}, followed by R_{pref}, and on the right ${\mathsf{R}_{\text{pref}}}^2$.

▶ **Corollary 3.7.** *The topology* $\mathsf{lfp}_\tau.\mathsf{R}_{\text{pref}}(\tau)$ *is not Noetherian.*

The prefix topology is not Noetherian, even when starting from a finite alphabet. However, we claimed in Section 1 that there is a natural generalisation of the subword embedding to topological spaces which is Noetherian. Before introducing this topology, let us write $[U_1, \dots, U_n]$ as a shorthand notation for the set $\Sigma^* U_1 \Sigma^* \dots \Sigma^* U_n \Sigma^*$.

▶ **Definition 3.8** (Subword topology [12, Definition 9.7.26])**.** *Given a topological space* (Σ, τ)*, the space* Σ^* *of finite words over* Σ *can be endowed with the* subword topology*, generated by the open sets* $[U_1, \dots, U_n]$ *when* $U_i \in \tau$*.*

The *topological Higman lemma* [12, Theorem 9.7.33] states that the subword topology over Σ^* is Noetherian if and only if Σ is Noetherian. Let us now reverse engineer a refinement function whose least fixed point is the subword topology.

▶ **Definition 3.9.** *Let* (Σ, θ) *be a topological space. Let* $\mathsf{E}_{\text{words}}$ *be defined as mapping a topology* τ *over* Σ^* *to the topology generated by the following sets:* $\uparrow_{\leq_*} UV$ *for* $U, V \in \tau$*; and* $\uparrow_{\leq_*} W$*, for* $W \in \theta$*.*

▶ **Lemma 3.10.** *Let* (Σ, θ) *be a topological space. The subword topology over* Σ^* *is the least fixed point of* $\mathsf{E}_{\text{words}}$*.*

Proof. First, we notice that the subword topology is stable under $\mathsf{E}_{\text{words}}$. Then, we prove by induction on n shows that $[U_1, \dots, U_n]$ is open in the least fixed point of $\mathsf{E}_{\text{words}}$. ◀

It is an easy check that the subword topology over Σ^* is the least fixed point of $\mathsf{E}_{\text{words}}$. In order to show that $\mathsf{E}_{\text{words}}$ is a refinement function, we first claim that the two parts of the topology can be dealt with separately.

▶ **Lemma 3.11** ([12, Proposition 9.7.18])**.** *If* $(\mathcal{X}, \tau)$ *and* $(\mathcal{X}, \tau')$ *are Noetherian, then* $\mathcal{X}$ *endowed the topology generated by* $\tau \cup \tau'$ *is Noetherian.*

Proof. The space $\mathcal{X}$ endowed with the topology generated by $\tau \cup \tau'$ is a quotient of $(\mathcal{X}, \tau) + (\mathcal{X}, \tau')$. Therefore, it is Noetherian [12, Proposition 9.7.18]. ◀

▶ **Lemma 3.12.** *Let* (Σ, θ) *be a Noetherian topological space. The map* $\mathsf{E}_{\text{words}}$ *is a refinement function over* Σ*.*

Proof. We leave the monotonicity of $\mathsf{E}_{\text{words}}$ as an exercice and focus on the proof that $\mathsf{E}_{\text{words}}(\tau)$ is Noetherian, whenever τ is. Thanks to Lemma 3.11, it suffices to prove that the topology generated by the sets $\uparrow_{\leq_*} UV$ (U, V open in τ), and the topology generated by the sets $\uparrow_{\leq_*} W$ (W open in θ) are Noetherian.

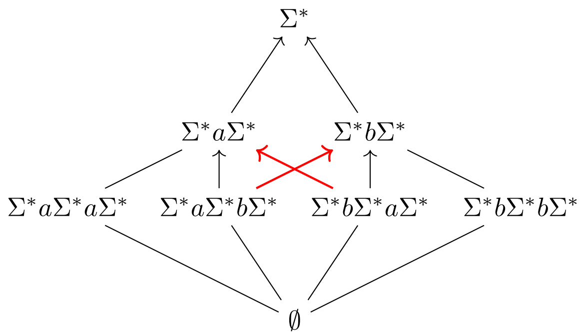

Figure 2 The topology $\mathsf{E}_{\text{words}}{}^2(\tau_{\text{triv}})$, with bold red arrows for the inclusions that were not present between the "analogous sets" in $\mathsf{R}_{\text{pref}}{}^2(\tau_{\text{triv}})$.

Let $(\uparrow_{\leq_*} U_i V_i)_{i \in \mathbb{N}}$ be a sequence of open sets. Because Noetherian topologies are closed under products (Table 1), the sequence $(\bigcup_{i \leq k} U_i \times V_i)_{k \in \mathbb{N}}$ is asymptotically constant. Hence, the sequence $\bigcup_{i \leq k} \uparrow_{\leq_*} U_i V_i$ also is.

Let $\uparrow_{\leq_*} W_i$ be a sequence of open sets. Because θ is Noetherian, the sequence $\bigcup_{i \leq k} W_i$ is asymptotically constant, hence so is the sequence $\bigcup_{i \leq k} \uparrow_{\leq_*} W_i$. ◀

We have designed two refinement functions R_{pref} and $\mathsf{E}_{\text{words}}$ over Σ^*. The least fixed point of the former is not Noetherian, as opposed to the least fixed point of the latter. We have depicted the result of iterating $\mathsf{E}_{\text{words}}$ twice over the trivial topology in Figure 2. As opposed to R_{pref}, the "diagonal" elements are comparable for inclusion. To further elaborate the difference between R_{pref} and $\mathsf{E}_{\text{words}}$, let us compare their behaviour with respect to subsets of Σ^*.

Let $V := a\Sigma^*$, which is a closed subset of $(\Sigma^*, \mathsf{R}_{\text{pref}}(\tau_{\text{triv}}))$. When endowing V with the topology induced by $\mathsf{R}_{\text{pref}}(\tau_{\text{triv}})$, we obtain the space (V, τ_{triv}). When endowing V with the topology induced by $\mathsf{R}_{\text{pref}}{}^2(\tau_{\text{triv}})$, we obtain a space $(V, \{\emptyset, aa\Sigma^*, ab\Sigma^*, V\})$. However, if one considers V as a topological space itself, then applying R_{pref} over (V, τ_{triv}) leads to the open sets $\{\emptyset, aa\Sigma^*, V\}$, which is a different topology.

Let $W := \Sigma^* a \Sigma^*$, which is a closed in $\mathsf{E}_{\text{words}}(\tau_{\text{triv}})$. As for V, the topology induced on W by $\mathsf{E}_{\text{words}}(\tau_{\text{triv}})$ is the trivial topology. However, when considering (W, τ_{triv}) as a topological space, we obtain the same topology over W whether we build the topology induced by $\mathsf{E}_{\text{words}}{}^2(\tau_{\text{triv}})$, or apply $\mathsf{E}_{\text{words}}$ to W itself.

3.2 Well-behaved refinement functions

As hinted in the previous section, the behaviour of the refinement function with respect to subsets will act as a sufficient condition to separate the well-behaved ones from the others. In order to make the idea of computing the refinement function directly over a subset precise, we will replace a subset with the induced topology by a "restricted" topology over the whole space.

▶ **Definition 3.13.** *Let $(\mathcal{X}, \tau)$ be a topological space and H be a closed subset of $\mathcal{X}$. Define the* subset restriction *$\tau | H$ to be the topology generated by the opens $U \cap H$ where U ranges over τ.*

Let $\mathcal{X}$ be a topological space, and H be a proper closed subset of $\mathcal{X}$. The space $\mathcal{X}$ endowed with $\tau | H$ has a lattice of open sets that is isomorphic to the one of the space H

endowed with the topology induced by τ, except for the entire space $\mathcal{X}$ itself. Beware that, the two spaces are in general not homeomorphic.

▶ **Example 3.14.** Let $\mathbb{R}$ be endowed with the usual metric topology. The set $\{a\}$ is a closed set. The induced topology over $\{a\}$ is $\{\emptyset, \{a\}\}$. The subset restriction of the topology to $\{a\}$ is $\tau_a := \{\emptyset, \{a\}, \mathbb{R}\}$. Clearly, $(\mathbb{R}, \tau_a)$ and $(\{a\}, \tau_{\text{triv}})$ are not homeomorphic.

In order to build intuition, let us consider the special case of an Alexandroff topology over X and compute the specialisation preorder of $\tau|H$, where H is a downwards closed set.

▶ **Lemma 3.15.** *Let $\tau = \mathsf{alex}(\leq)$ over a set X, and $x, y \in X$. Then, $x \leq_{\tau|H} y$ if and only if $x \leq_\tau y \in H$ or $x \notin H$.*

Proof. Let us write $\uparrow F$ for the set of points that are above F for $\leq$, and $\uparrow x$ as a shorthand notation for $\uparrow \{x\}$, the set of points above x. Let us now unpack the definition of $x \leq_{\tau|H} y$.

$$\begin{aligned} x \leq_{\tau|H} y &\iff \forall U \in \tau|H, x \in U \Rightarrow y \in U \\ &\iff \forall U \in \tau, x \in U \cap H \Rightarrow y \in U \cap H \end{aligned}$$

Let $x \leq_{\tau|H} y$. If $x \in H$, then for every open set $U \in \tau$, $x \in U \cap H$, hence $y \in U \cap H$. As a consequence, $x \leq_\tau y$ and both belong to H.

Conversely, assume that $x \notin H$, then $x \in U \cap H \implies y \in U \cap H$ vacuously for every $U \in \tau$, hence $x \leq_{\tau|H} y$. Whenever, $x \leq_\tau y \in H$, then $x \notin H$ implies $y \notin H$, which is absurd. Therefore, $x \leq_{\tau|H} y$. ◀

▶ **Definition 3.16.** *A* topology expander *is a refinement function* E *that satisfies the following extra property: for every Noetherian topology τ satisfying $\tau \subseteq \mathsf{E}(\tau)$, for all closed set H in τ, $\mathsf{E}(\tau)|H = \mathsf{E}(\tau|H)|H$. We say that* E *respects subsets.*

▶ Note 3.17. In Definition 3.16, the equality can be replaced by: H is closed in $\mathsf{E}(\tau|H)$ and $\mathsf{E}(\tau)|H \subseteq \mathsf{E}(\tau|H)|H$.

As proven at the end of Section 3.1, R_{pref} fails to be a topology expander. Let us quickly prove that $\mathsf{E}_{\text{words}}$ is a topology expander.

▶ **Lemma 3.18.** *Let (Σ, θ) be a Noetherian space. Then* $\mathsf{E}_{\text{words}}$ *is a topology expander.*

Proof. We have proven in Lemma 3.12 that $\mathsf{E}_{\text{words}}$ is a refinement function. Let us now prove that it respects subsets.

Let τ be a Noetherian topology over Σ^*, such that $\tau \subseteq \mathsf{E}_{\text{words}}(\tau)$. Let H be a closed subset of (Σ^*, τ). Notice that as H is closed in τ, and since $\tau \subseteq \mathsf{E}_{\text{words}}(\tau)$, H is downwards closed for $\leq_*$. As a consequence, $(\uparrow_{\leq_*} UV) \cap H = (\uparrow_{\leq_*} (U \cap H)(V \cap H)) \cap H$. Similarly, $(\uparrow_{\leq_*} W) \cap H = (\uparrow_{\leq_*} (W \cap H)) \cap H$. Hence, $\mathsf{E}_{\text{words}}(\tau)|H \subseteq \mathsf{E}_{\text{words}}(\tau|H)|H$. ◀

3.3 Iterating Expanders

Our goal is now to prove that topology expanders are refinement functions that can be safely iterated. For that, let us first define precisely what "iterating transfinitely" a refinement function means.

▶ **Definition 3.19.** *Let $(\mathcal{X}, \tau)$ be a topological space, and* E *be a topology expander. The* limit topology $\mathsf{E}^\alpha(\tau)$ *is defined as: τ when $\alpha = 0$, $\mathsf{E}(\mathsf{E}^\beta(\tau))$ when $\alpha = \beta + 1$, and as the join of the topologies $\mathsf{E}^\beta(\tau)$ for all $\beta < \alpha$, when α is a limit ordinal.*

We devote the rest of this section to proving our main theorem, which immediately implies that least fixed points of topology expanders are Noetherian. Notice that the theorem is trivial whenever α is a successor ordinal.

▶ **Proposition 3.20.** *Let α be an ordinal, τ be a topology, and E be a topology expander. If $\mathsf{E}^{\beta}(\tau)$ is Noetherian for all $\beta < \alpha$, and $\tau \subseteq \mathsf{E}(\tau)$, then $\mathsf{E}^{\alpha}(\tau)$ is Noetherian.*

▶ **Theorem 3.21** (Main Result)**.** *Let X be a set and E be a topology expander. The least fixed point of E is a Noetherian topology over X.*

3.3.1 The topological minimal bad sequence argument.

In order to define what a minimal bad sequence is, we first introduce a well-founded partial ordering over the elements of $\mathsf{E}^{\alpha}(\tau)$. With an open set $U \in \mathsf{E}^{\alpha}(\tau)$, we associate a depth $\mathsf{depth}(U)$, defined as the smallest ordinal $\beta \leq \alpha$ such that $U \in \mathsf{E}^{\beta}(\tau)$. We then define $U \trianglelefteq V$ to hold whenever $\mathsf{depth}(U) \leq \mathsf{depth}(V)$, and $U \triangleleft V$ whenever $\mathsf{depth}(U) < \mathsf{depth}(V)$. It is an easy check that this is a well-founded partial order over $\mathsf{E}^{\alpha}(\tau)$.

As a first step towards proving that $\mathsf{E}^{\alpha}(\tau)$ is Noetherian for a limit ordinal α, we first reduce the problem to opens of depth strictly less than α itself.

▶ **Lemma 3.22.** *Let α be a limit ordinal, and E be a topology expander. The topology $\mathsf{E}^{\alpha}(\tau)$ has a subbasis of elements of depth strictly below α.*

Proof. By definition of the limit topology. ◀

Let us recall the notion of topological bad sequence designed by Goubault-Larrecq [12, Lemma 9.7.31] in the proof of the Topological Kruskal Theorem, adapted to our ordering of subbasic open sets. This notion of bad sequence is tailored to mimic the notion of good sequences and bad sequences in well-quasi-orderings.

▶ **Definition 3.23.** *Let $(\mathcal{X}, \tau)$ be a topological space. A sequence $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ of open sets is* good *if there exists $i \in \mathbb{N}$ such that $U_i \subseteq \bigcup_{j<i} U_j$. A sequence that is not good is called* bad*.*

▶ **Lemma 3.24.** *Let α be a limit ordinal, and E be a topology expander such that $\mathsf{E}^{\alpha}(\tau)$ is not Noetherian. Then, there exists a bad sequence $\mathcal{U}$ of opens in $\mathsf{E}^{\alpha}(\tau)$ of depth less than α that is lexicographically minimal for $\trianglelefteq$. Such a sequence is called* minimal bad*.*

Proof. Assume that $\mathsf{E}^{\alpha}(\tau)$ is not Noetherian. There exists a sequence $(U_i)_{i \in \mathbb{N}}$ of subbasic open sets that is bad and lexicographically minimal with respect to $\trianglelefteq$ [12, Lemma 9.7.31]. ◀

We deduce that in a limit topology, minimal bad sequences are not allowed to use opens of arbitrary depth.

▶ **Lemma 3.25.** *Let α be a limit ordinal, τ be a topology and E be a topology expander such that $\mathsf{E}^{\beta}(\tau)$ is Noetherian for all $\beta < \alpha$. Assume that $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ is a minimal bad sequence of $\mathsf{E}^{\alpha}(\tau)$. Then, for every $i \in \mathbb{N}$, $\mathsf{depth}(U_i)$ is either 0 or a successor ordinal.*

Proof. Assume by contradiction that there exists $i \in \mathbb{N}$ such that $\mathsf{depth}(U_i)$ is a limit ordinal d_i. This proves that U_i is obtained as a union of open sets in $\mathsf{E}^{\beta}(\tau)$ for $\beta < d_i$. Since $\mathsf{E}^{d_i}(\tau)$ is Noetherian, one can define U_i as a finite union of open sets of depth less than d_i. As a consequence, $\mathsf{depth}(U_i) < d_i$, which is absurd. ◀

▶ **Definition 3.26.** *Let α be an ordinal, τ be a topology, E be a topology expander such that $\tau \subseteq \mathsf{E}(\tau)$, and let $U \in \mathsf{E}^\alpha(\tau)$. The topology $\mathsf{Down}(U)$ is generated by the open sets V such that $V \subseteq U$, where V ranges over $\mathsf{E}^\alpha(\tau)$.*

▶ **Lemma 3.27.** *Let α be an ordinal, τ be a topology, E be a topology expander such that $\tau \subseteq \mathsf{E}(\tau)$, and let $U \in \mathsf{E}^\alpha(\tau)$. If $\mathsf{depth}(U) = \gamma + 1$, and $\mathsf{E}^\gamma(\tau)$ is Noetherian, then $U \in \mathsf{E}(\mathsf{Down}(U))$.*

Proof. Let $U \in \mathsf{E}^{\gamma+1}(\tau) = \mathsf{E}(\mathsf{E}^\gamma(\tau))$. By definition, an open set $V \in \mathsf{E}^\alpha(\tau)$ satifies $\mathsf{depth}(V) < \mathsf{depth}(U)$ if and only if it belongs to $\mathsf{E}^\gamma(\tau)$. As a consequence, $\mathsf{E}^\gamma(\tau) = \mathsf{Down}(U)$, and $U \in \mathsf{E}(\mathsf{Down}(U))$. ◀

If $\mathcal{U}$ is a minimal bad sequence in $(X, \mathsf{E}^\alpha(\tau))$, then $U_i \not\subseteq \bigcup_{j<i} U_j := V_i$, i.e., $U_i \cap V_i^c \neq \emptyset$. We can now use our subset restriction operator to devise a topology associated to this minimal bad sequence. Noticing that $H_i := V_i^c$ is a closed set in $\mathsf{E}^\alpha(\tau)$, we can build the subset restriction $\mathsf{Down}(U_i)|H_i$.

▶ **Definition 3.28.** *Let α be an ordinal, τ be a topology, E be a topology expander such that $\tau \subseteq \mathsf{E}(\tau)$, and let $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ be a minimal bad sequence in $\mathsf{E}^\alpha(\tau)$. Then, the* minimal topology *$\mathcal{U}(\mathsf{E}^\alpha(\tau))$ is generated by $\bigcup_{i\in\mathbb{N}} \mathsf{Down}(U_i)|H_i$, where $H_i := (\bigcup_{j<i} U_j)^c$.*

▶ **Lemma 3.29.** *Let α be an ordinal, τ be a topology, E be a topology expander such that $\tau \subseteq \mathsf{E}(\tau)$, and let $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ be a minimal bad sequence in $\mathsf{E}^\alpha(\tau)$. Then, the minimal topology $\mathcal{U}(\mathsf{E}^\alpha(\tau))$ is Noetherian.*

Proof. Assume by contradiction that $\mathcal{U}(\mathsf{E}^\alpha(\tau))$ is not Noetherian. Let us define V_i as $\bigcup_{j<i} U_j$, and H_i as V_i^c.

Thanks to [12, Lemma 9.7.15] there exists a bad sequence $\mathcal{W} := (W_i)_{i\in\mathbb{N}}$ of subbasic elements of $\mathcal{U}(\mathsf{E}^\alpha(\tau))$. By definition, W_i is in some $\mathsf{Down}(U_j)|H_j$. Let us select a mapping $\rho\colon \mathbb{N} \to \mathbb{N}$, such that $W_i \in \mathsf{Down}(U_{\rho(i)})|H_{\rho(i)}$. In practice, this amounts to the existence of an open $T_{\rho(i)}$, such that $T_{\rho(i)} \triangleleft U_{\rho(i)}$, $T_{\rho(i)} \subseteq U_{\rho(i)}$, and $W_i = T_{\rho(i)} \setminus V_{\rho(i)}$. Without loss of generality we assume that ρ is monotonic.

Let us build the sequence $\mathcal{Y}$ defined by $Y_i := U_i$ if $i < \rho(0)$ and $Y_i := T_{\rho(i)}$ otherwise. This is a sequence of open sets in $\mathsf{E}^\alpha(\tau)$ that is lexicographically smaller than $\mathcal{U}$, hence $\mathcal{Y}$ is a good sequence: there exists $i \in \mathbb{N}$ such that $Y_i \subseteq \bigcup_{j<i} Y_j$.

- If $i < \rho(0)$, then $U_i \subseteq \bigcup_{j<i} U_j$ contradicting that $\mathcal{U}$ is *bad*.
- If $i \geq \rho(0)$, let us write $Y_i = T_{\rho(i)} \subseteq \bigcup_{j<\rho(0)} U_j \cup \bigcup_{j<i} T_{\rho(j)}$. By taking the intersection with $H_{\rho(i)}$, we obtain $W_i \subseteq \bigcup_{j<i} W_j$, contradicting the fact that $\mathcal{W}$ is a bad sequence. ◀

We are now ready to leverage our knowledge of minimal topologies associated with minimal bad sequences to carry on the proof of our main theorem.

▶ **Proposition 3.20.** *Let α be an ordinal, τ be a topology, and E be a topology expander. If $\mathsf{E}^\beta(\tau)$ is Noetherian for all $\beta < \alpha$, and $\tau \subseteq \mathsf{E}(\tau)$, then $\mathsf{E}^\alpha(\tau)$ is Noetherian.*

Proof. If α is a sucessor ordinal, then $\alpha = \beta + 1$ and $\mathsf{E}^\alpha(\tau) = \mathsf{E}(\mathsf{E}^\beta(\tau))$. Because E respects Noetherian topologies, we immediately conclude that $\mathsf{E}^\alpha(\tau)$ is Noetherian. We are therefore only interested in the case where α is a limit ordinal.

Assume by contradiction that $\mathsf{E}^\alpha(\tau)$ is not Noetherian, using Lemma 3.24 there exists a minimal bad sequence $\mathcal{U} := (U_i)_{i\in\mathbb{N}}$. Let us write $d_i := \mathsf{depth}(U_i) < \alpha$. Thanks to Lemma 3.25, d_i is either 0 or a successor ordinal.

Because $\mathsf{E}^{\beta}(\tau)$ is Noetherian for $\beta < \alpha$, there are finitely many opens U_i at depth β for every ordinal $\beta < \alpha$. Indeed, if they were infinitely many, one would extract an infinite bad sequence of opens in $\mathsf{E}^{\beta}(\tau)$, which is absurd.

Furthermore, the sequence $(d_i)_{i\in\mathbb{N}}$ must be monotonic, otherwise $\mathcal{U}$ would not be lexicographically minimal. We can therefore construct a strictly increasing map $\rho\colon \mathbb{N} \to \mathbb{N}$ such that $0 < \mathsf{depth}(U_{\rho(j)})$ and $\mathsf{depth}(U_i) < \mathsf{depth}(U_{\rho(j)})$ whenever $0 \leq i < \rho(j)$.

Let us consider some $i = \rho(n)$ for some $n \in \mathbb{N}$. Let us write $V_i := \bigcup_{j<i} U_j$, and $H_i := X \setminus V_i$. The set V_i is open in $\mathsf{Down}(U_i)$ by construction of ρ, hence H_i is closed in $\mathsf{Down}(U_i)$. As E is a topology expander, we derive the following inclusions:

$$\begin{aligned}\mathsf{E}(\mathsf{Down}(U_i))|H_i &\subseteq \mathsf{E}(\mathsf{Down}(U_i)|H_i)|H_i \\ &\subseteq \mathsf{E}(\mathcal{U}(\mathsf{E}^{\alpha}(\tau)))|H_i\end{aligned}$$

Recall that $U_i \in \mathsf{E}(\mathsf{Down}(U_i))$ thanks to Lemma 3.27. As a consequence, $U_i \setminus V_i = W_i \setminus V_i$ for some open set W_i in $\mathsf{E}(\mathcal{U}(\mathsf{E}^{\alpha}(\tau)))$. Thanks to Lemma 3.29, and preservation of Noetherian topologies through topology expanders, the latter is a Noetherian topology. Therefore, $(W_{\rho(i)})_{i\in\mathbb{N}}$ is a good sequence. This provides an $i \in \mathbb{N}$ such that $W_{\rho(i)} \subseteq \bigcup_{\rho(j)<\rho(i)} W_{\rho(j)}$. In particular,

$$\begin{aligned}U_{\rho(i)} \setminus V_{\rho(i)} = W_{\rho(i)} \setminus V_{\rho(i)} &\subseteq \bigcup_{\rho(j)<\rho(i)} W_{\rho(j)} \setminus V_{\rho(i)} \subseteq \bigcup_{\rho(j)<\rho(i)} W_{\rho(j)} \setminus V_{\rho(j)} \\ &\subseteq \bigcup_{\rho(j)<\rho(i)} U_{\rho(j)} \setminus V_{\rho(j)} \subseteq \bigcup_{j<\rho(i)} U_j = V_{\rho(i)}\end{aligned}$$

This proves that $U_{\rho(i)} \subseteq V_{\rho(i)}$, i.e. that $U_{\rho(i)} \subseteq \bigcup_{j<\rho(i)} U_j$. Finally, this contradicts the fact that $\mathcal{U}$ is bad. ◀

We have effectively proven that being well-behaved with respect to closed subspaces is enough to consider least fixed points of refinement functions. This behaviour should become clearer in the upcoming sections, where we illustrate how this property can be ensured both in the case of Noetherian spaces and well-quasi-orderings.

4 Applications of Topology Expanders

We now briefly explore topologies that can be proven to be Noetherian using Theorem 3.21. It should not be surprising that both the topological Higman lemma and the topological Kruskal theorem fit in the framework of topology expanders, as both were already proven using a minimal bad sequence argument. However, we will proceed to extend the use of topology expander to spaces for which the original proof did not use a minimal bad sequence argument, and illustrate how they can easily be used to define new Noetherian topologies.

4.1 Finite words and finite trees

As a first example, we can easily recover the *topological Higman lemma* [12, Theorem 9.7.33] because the subword topology is the least fixed point of $\mathsf{E}_{\mathrm{words}}$ (Lemma 3.10), which is a topology expander (Lemma 3.18).

It does not require much effort to generalise this proof scheme to the case of the *topological Kruskal theorem* [12, Theorem 9.7.46]. As a shorthand notation, let us write $t \in \diamond U\langle V\rangle$ whenever there exists a subtree t' of t whose root is labelled by an element of U and whose list of children belongs to V.

▶ **Definition 4.1** ([12, Definition 9.7.39]). *Let (Σ, θ) be a topological space. The space $\mathsf{T}(\Sigma)$ of finite trees over Σ can be endowed with the* tree topology*, the coarsest topology such that $\diamond U\langle V\rangle$ is open whenever U is an open set of Σ, and V is an open set of $\mathsf{T}(\Sigma)^*$ in its subword topology.*

▶ **Definition 4.2.** *Let (Σ, θ) be a topological space. Let E_{tree} be the function that maps a topology τ to the topology generated by the sets $\uparrow_{\leq_{\mathsf{tree}}} U\langle V\rangle$, for U open in θ, V open in $\mathsf{T}(\Sigma)^*$ with the subword topology of τ.*

▶ **Lemma 4.3.** *The tree topology is the least fixed point of E_{tree}, which is a topology expander.*

Proof. The proof is follows the same pattern as for the subword topology. The only technical part is to notice that a downwards closed set H for $\leq_{\mathsf{tree}}$ satisfies $(\uparrow_{\leq_{\mathsf{tree}}} U\langle V\rangle) \cap H = (\uparrow_{\leq_{\mathsf{tree}}} U\langle [V_1 \cap H, \ldots, V_n \cap H]\rangle) \cap H$, whenever $V = [V_1, \ldots, V_n]$. ◀

▶ **Corollary 4.4.** *The tree topology is Noetherian.*

4.2 Ordinal words

Let us now demonstrate how Theorem 3.21 can be applied over spaces for which the original proof of Noetheriannes did not use a minimal bad sequence argument. For that, let us consider $\Sigma^{<\alpha}$ the set of words of ordinal length less than α, where α is a fixed ordinal. Since $\leq_*$ is in general not a wqo on $\Sigma^{<\alpha}$ when $\leq$ is wqo on Σ, this also provides an example of a topological minimal bad sequence argument that has no counterpart in the realm of wqos.

▶ **Definition 4.5** ([15]). *Let (Σ, θ) be a topological space. The* ordinal subword topology *over $\Sigma^{<\alpha}$ is the topology generated by the closed sets $F_1^{<\beta_1} \cdots F_n^{<\beta_n}$, for $n \in \mathbb{N}$, for F_i closed in θ, and where $F^{<\beta}$ is the set of words of length less than β with all of their letters in F.*

The ordinal subword topology is Noetherian [15], but the proof is quite technical and relies on the in-depth study of the possible inclusions between the subbasic closed sets. Before defining a suitable topology expander, given an ordinal β and a set $U \subseteq \Sigma^{<\alpha}$, let us write $w \in \beta \triangleright U$ if and only if $w_{>\gamma} \in U$ for all $0 \leq \gamma < \beta$.

▶ **Definition 4.6.** *Let (Σ, θ) be a topological space, and α be an ordinal. The function $\mathsf{E}_{\alpha\text{-words}}$ maps a topology τ to the topology generated by the following sets: $\uparrow_{\leq_*} UV$ for U, V opens in τ; $\uparrow_{\leq_*} \beta \triangleright U$, for U open in τ, $\beta \leq \alpha$; $\uparrow_{\leq_*} W$, for W open in θ.*

▶ **Lemma 4.7.** *Given a Noetherian space (Σ, θ), and an ordinal α. The map $\mathsf{E}_{\alpha\text{-words}}$ is a topology expander, whose least fixed point contains the ordinal subword topology.*

Proof. It is obvious that $\mathsf{E}_{\alpha\text{-words}}$ is monotone. Moreover, the closed sets H in $\mathsf{E}_{\alpha\text{-words}}(\tau)$ are downwards closed with respect to $\leq_*$. As a consequence, $(\uparrow_{\leq_*} UV) \cap H = (\uparrow_{\leq_*} (U \cap H)(V \cap H)) \cap H$, $(\uparrow_{\leq_*} W) \cap H = (\uparrow_{\leq_*} (W \cap H)) \cap H$, and $(\uparrow_{\leq_*} \beta \triangleright U) \cap H = (\uparrow_{\leq_*} \beta \triangleright (U \cap H)) \cap H$. Hence, $\mathsf{E}_{\alpha\text{-words}}$ respects subsets. To conclude that $\mathsf{E}_{\alpha\text{-words}}$ is a topology expander, it remains to prove that it preserves Noetherian topologies.

▷ Claim 4.8. Let τ be a Noetherian topology. Then $\mathsf{E}_{\alpha\text{-words}}(\tau)$ is Noetherian.

Proof. As a consequence of Lemma 3.12, the topology generated by the sets $\uparrow_{\leq_*} UV$, and $\uparrow_{\leq_*} W$ is Noetherian. Therefore, it suffices to check that the topology generated by the sets $\uparrow_{\leq_*} \beta \triangleright U$ is Noetherian to conclude that $\mathsf{E}_{\alpha\text{-words}}(\tau)$ is too.

For that, consider a bad sequence $\beta_i \triangleright U_i$ of open sets, indexed by $\mathbb{N}$. Because for all i, $\beta_i < \alpha + 1$, we can extract our sequence so that $\beta_i \leq \beta_j$ when $i \leq j$. The extracted sequence is still bad. Because τ is Noetherian, there exists $i \in \mathbb{N}$ such that $U_i \subseteq \bigcup_{j<i} U_j$. Let us now conclude that $\beta_i \triangleright U_i \subseteq \bigcup_{j<i} \beta_j \triangleright U_j$, which is in contradiction with the fact that the sequence is bad.

Let $w \in \beta_i \triangleright U_i$, and assume by contradiction that for all $j < i$, there exists a $\gamma_j < \beta_j \leq \beta_i$ such that $w_{>\gamma_j} \notin U_j$. Let $\gamma := \max_{j<i} \gamma_j < \beta_i$. The word $w_{>\gamma}$ does not belong to U_j for $j < i$, because U_j is upwards closed for $\leq_*$. As a consequence, $w_{>\gamma} \notin \bigcup_{j<i} U_j$. However, $w_{>\gamma} \in U_i$, which is absurd. ◁

We now have to check that every open set in the ordinal subword topology is open in the least fixed point of $\mathsf{E}_{\alpha\text{-words}}$. We prove by induction over n that a product $F_1^{<\beta_1} \dots F_n^{<\beta_n}$ has a complement that is open.

Empty product this is the whole space.

$\boldsymbol{P := F^{<\beta} P'}$ By induction hypothesis, P'^c is an open U in the least fixed point topology. Let us prove that $P^c = A \cup B$, where $A := \uparrow_{\leq_*} \{av \colon u \notin F \wedge av \in U\}$, and $B := \uparrow_{\leq_*} (\beta \triangleright U)$.

▷ Claim 4.9. $P^c \subseteq A \cup B$.

Proof. Let $w \notin P$ and distinguish two cases.

- Either there exists a smallest $\gamma < \beta$ such that $w_\gamma \notin F$. In which case $w = w_{<\gamma} w_\gamma w_{>\gamma}$. Since $\gamma < \beta$, $w_{\leq\gamma} \in F^{<\beta}$, hence $w_{>\gamma} \in U$ because $w \notin P$. As a consequence, $w \in A$.
- Or $w_\gamma \in F$ for every $\gamma < \beta$. However, this proves that $w_{>\gamma} \in U$ for every $\gamma < \beta$, which means that $w \in B$. ◁

▷ Claim 4.10. $A \subseteq P^c$.

Proof. Because P is downwards closed for $\leq_*$, it suffices to check that every word av with $a \notin F$ and $av \in U$ lies in P^c.

Assume by contradiction that $av \in P$, then $av = u_1 u_2$ with $u_1 \in F^{<\beta}$ and $u_2 \in P'$. Because $a \notin F$, this proves that u_1 is the empty word, and that $u_2 = w \in P'$. This is absurd because $w \in U = (P')^c$. ◁

▷ Claim 4.11. $B \subseteq P^c$.

Proof. Because P is downwards closed for $\leq_*$ it suffices to check that every word $w \in \beta \triangleright U$ lies in P^c.

Assume by contradiction that such a word w is in P. One can write $w = uv$ with $u \in F^{<\beta}$ and $v \in P'$. However, $|u| = \gamma < \beta$, and $\gamma + 1 < \beta$ because β is a limit ordinal. Therefore, $v = w_{>\gamma} \in U = (P')^c$ which is absurd. ◁

▷ Claim 4.12. A and B are open in the least fixed point of $\mathsf{E}_{\alpha\text{-words}}$.

Proof. The set B is open because U is open. Let us prove by induction that whenever U is open and F is closed in θ, the set $F \rtimes U$ defined as $\uparrow_{\leq_*} \{av \colon a \notin F, av \in U\}$ is open. It is easy to check that $F \rtimes (\uparrow_{\leq_*} W) = \uparrow_{\leq_*} (W \cap F^c) \cup \uparrow_{\leq_*} F^c W$. Moreover, $F \rtimes (\uparrow_{\leq_*} UV) = \uparrow_{\leq_*} (F \rtimes U) V$. Finally, for $\beta \geq 1$, $F \rtimes (\uparrow_{\leq_*} \beta \triangleright U) = \uparrow_{\leq_*} F^c (\beta' \triangleright U)$ with $\beta' = \beta$ if β is limit, and $\beta' = \gamma$ if $\beta = \gamma + 1$. ◁

We have proven that P^c is open. ◀

▶ **Corollary 4.13.** *The ordinal subword topology is Noetherian.*

4.3 Ordinal branching trees

As an example of a new Noetherian topology derived using Theorem 3.21, we will consider *α-branching trees* $\mathsf{T}^{<\alpha}(\Sigma)$, i.e., the least fixed point of the constructor $X \mapsto \mathbf{1} + \Sigma \times X^{<\alpha}$ where α is a given ordinal.

▶ **Definition 4.14.** *Let (Σ, θ) be a Noetherian space. The* ordinal tree topology *over α-branching trees is the least fixed point of $\mathsf{E}_{\alpha\text{-trees}}$, mapping a topology τ to the topology generated by the sets $\uparrow_{\leq_{\mathsf{tree}}} U\langle V\rangle$, where $U \in \theta$, V is open in $(\mathsf{T}^{<\alpha}(\Sigma))^{<\alpha}$ with the ordinal subword topology, and $U\langle V\rangle$ is the set of trees whose root is labelled by an element of U and list of children belongs to V.*

▶ **Theorem 4.15.** *The α-branching trees endowed with the ordinal tree topology forms a Noetherian space.*

Proof. It suffices to prove that $\mathsf{E}_{\alpha\text{-trees}}$ is a topology expander. It is clear that $\mathsf{E}_{\alpha\text{-trees}}$ is monotone, and a closed set of $\mathsf{E}_{\alpha\text{-trees}}(\tau)$ is always downwards closed for $\leq_{\mathsf{tree}}$. As a consequence, if $\tau \subseteq \mathsf{E}_{\alpha\text{-trees}}(\tau)$ and H is closed in τ, $t \in V := (\uparrow_{\leq_{\mathsf{tree}}} U\langle V\rangle) \cap H$ if and only if $t \in H$ and every children of t belongs to H. Therefore, $(\uparrow_{\leq_{\mathsf{tree}}} U\langle V\rangle) \cap H = (\uparrow_{\leq_{\mathsf{tree}}} U\langle V \cap H^{<\alpha}\rangle) \cap H$. Notice that $H^{<\alpha} \cap V$ is an open of the ordinal subword topology over $\tau|H$. As a consequence, $V \cap H \in \mathsf{E}_{\alpha\text{-trees}}(\tau|H)|H$.

Let us now check that $\mathsf{E}_{\alpha\text{-trees}}$ presreves Noetherian topologies. Let $W_i :=\uparrow_{\leq_{\mathsf{tree}}} U_i\langle V_i\rangle$ be a $\mathbb{N}$-indexed sequence of open sets in $\mathsf{E}_{\alpha\text{-trees}}(\tau)$ where τ is Noetherian. The product of the topology θ and the ordinal subword topology over τ is Noetherian thanks to Table 1 and Lemma 4.7. Hence, there exists a $i \in \mathbb{N}$ such that $U_i \times V_i \subseteq \bigcup_{j<i} U_j \times V_j$. As a consequence, $W_i \subseteq \bigcup_{j<i} W_j$. We have proven that $\mathsf{E}_{\alpha\text{-trees}}(\tau)$ is Noetherian. ◀

At this point, we have proven that the framework of topology expanders allows to build non-trivial Noetherian spaces. We argue that this bears several advantages over ad-hoc proofs: (i) the ad-hoc proofs are often tedious and error prone [12, 13, 15] (ii) the verification that E is a topology expander on the other hand is quite simple (iii) the framework provides a good reason for which the desired topology is a sensible choice. However, this setting is not quite satisfactory yet, as we do not provide an automatic definition of the topology expander in the case of an inductively defined space.

5 Consequences on inductive definitions

So far, the process of constructing Noetherian spaces has been the following: first build a set of points, then compute a topology that is Noetherian as a least fixed point. In the case where the set of points itself is inductively defined (such as finite words or finite trees), the second step might seem redundant.

While inductive definitions are quite clear in the set theoretic interpretation, we are interested in quasi-orderings and topologies, for which the notion of least fixed-point has to be precised. To that purpose, let us now introduce some basic notions of category theory.

In this paper only three categories will appear, the category Set of sets and functions, the category Top of topological spaces and continuous maps, and the category Ord of quasi-ordered spaces and monotone maps. Using this language, a unary constructor G in the algebra of wqos defines an *endofunctor* from objects of the category Ord to objects of the category Ord preserving well-quasi-orderings.

In our study of Noetherian spaces (resp. well-quasi-orderings), we will often see constructors G' as first building a new set of structures, and then adapting the topology (resp.

ordering) to this new set. In categorical terms, we are interested in endofunctors G' that are U-lifts of endofunctors on Set.

▶ **Definition 5.1.** *An endofunctor G' of* Top *is a* lift *of an endofunctor G of* Set *if the following diagram commutes, where U is the forgetful functor*

$$\begin{array}{ccc} \mathsf{Top} & \xrightarrow{G'} & \mathsf{Top} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathsf{Set} & \xrightarrow{G} & \mathsf{Set} \end{array}$$

5.1 Divisibility Topologies over Analytic Functors

As noticed by Hasegawa [16] and Freund [8], usual orderings on words and trees can be derived from their least fixed point definitions. We will provide a similar construction for topological spaces. However, we will avoid as much as possible the use of complex machinery related to analytic functors, and use as a definition an equivalent characterisation given by Hasegawa [16, Theorem 1.6]. For an introduction to analytic functors and combinatorial species, we redirect the reader to Joyal [19].

▶ **Definition 5.2.** *Given* G *an endofunctor of* Set*, the* category of elements $\mathrm{el}(\mathsf{G})$ *has as objects pairs (E, a) with $a \in \mathsf{G}(E)$, and as morphisms between (E, a) and (E', a') maps $f : E \to E'$ such that $\mathsf{G}_f(a) = a'$.*

As an intuition to the unfamiliar reader, an element (E, a) in $\mathrm{el}(\mathsf{G})$ is a witness that a can be produced through G by using elements of E. Morphisms of elements are witnessing how relations between elements of $\mathsf{G}(E)$ and $\mathsf{G}(E')$ arise from relations between E and E'. As a way to define a "smallest" set of elements E such that a can be found in $\mathsf{G}(E)$, we rely on transitive objects.

▶ **Definition 5.3.** *A* transitive object *in a category $\mathcal{C}$ is an object X satisfying the following two conditions for every object A of $\mathcal{C}$: (a) $\mathrm{Hom}(X, A)$ is non-empty; (b) The right action of $\mathrm{Aut}(X)$ on $\mathrm{Hom}(X, A)$ by composition is transitive.*

Given an object A in a category $\mathcal{C}$, one can build the slice category $\mathcal{C}/A$ whose objects are elements of $\mathrm{Hom}(B, A)$ when B ranges over objects of $\mathcal{C}$ and morphisms between $c_1 \in \mathrm{Hom}(B_1, A)$ and $c_2 \in \mathrm{Hom}(B_2, A)$ are maps $f : B_1 \to B_2$ such that $c_2 \circ f = c_1$. This notion of slice category can be combined with the one of transitive object to build so-called "weak normal forms".

▶ **Definition 5.4.** *A* weak normal form *of an object A in a category $\mathcal{C}$ is a transitive object in $\mathcal{C}/A$.*

A category $\mathcal{C}$ has the *weak normal form property* whenever every object A has a weak normal form. We are now ready to formulate a definition of analytic functors through the existence of weak normal forms for objects in their category of elements.

▶ **Definition 5.5.** *An endofunctor* G *of* Set *is an* analytic functor *whenever its category of elements* $\mathrm{el}(\mathsf{G})$ *has the weak normal form property. Moreover; X is a finite set for every weak normal form $f \in \mathrm{Hom}((X, x), (Y, y))$ in $\mathrm{el}(\mathsf{G})/(Y, y)$.*

▶ **Example 5.6.** The functor mapping X to X^* is analytic, and the weak normal form of a word (X^*, w) is $(\mathrm{letters}(w), w)$ together with the canonical injection from $\mathrm{letters}(w)$ to X. In this specific case, the weak normal forms are in fact initial objects.

▶ **Example 5.7.** The functor mapping X to $X^{<\alpha}$ is not analytic when $\alpha \geq \omega$, because of the restriction that weak normal forms are defined using finite sets.

Let us now explain how these weak normal forms can be used to define a support associated to the analytic functor. Given an analytic functor G and an element (X, x) in $\mathrm{el}(\mathsf{G})$, there exists a weak normal form $f \in \mathrm{Hom}((Y, y), (X, x))$ in the slice category $\mathrm{el}(\mathsf{G})/(X, x)$. By definition, $f \colon Y \to X$ and $\mathsf{G}_f(y) = x$. We define $f(Y)$ as the support of x in X.

In turn, this construction of support allows building a substructure ordering on initial algebras $(\mu\mathsf{G}, \delta)$ of G: an element $a \in \mu\mathsf{G}$ is a child of an element $b \in \mu\mathsf{G}$ whenever $a = b$ or $a \in \mathrm{supp}(\delta^{-1}(b))$. The transitive closure of the children relation is called the *substructure ordering* on $\mu\mathsf{G}$, and written $\sqsubseteq$.

▶ **Example 5.8.** The substructure ordering on $\mu\mathsf{G}$ for $\mathsf{G}(X) := \mathbf{1} + \Sigma \times X$ is the suffix ordering of words.

As analytic functors induce a quasi-ordering on their initial algebras, it is natural to import this quasi-ordering when dealing with lifts of analytic functors in the category Ord. This follows the construction of Hasegawa [16, Definition 2.7], although this substructure ordering is implicitly built. Given a topology τ on $\mu\mathsf{G}$, one can build open sets as $\uparrow_{\sqsubseteq} U$ for $U \in \tau$. Open sets of this new topology are automatically upwards closed for $\sqsubseteq$.

▶ **Definition 5.9.** *Let $\mathsf{G}' \colon \mathsf{Top} \to \mathsf{Top}$ be a lifting of an analytic functor G, and $(\mu\mathsf{G}, \delta)$ an initial algebra of G. Moreover, we suppose that G' preserves inclusions. The* divisibility topology *over $\mu\mathsf{G}$ is the least fixed point of $\mathsf{E}_\Diamond$, mapping τ to the topology generated by $\left\{\uparrow_{\sqsubseteq}\delta(U) \colon U \text{ open in } \mathsf{G}'(\mu\mathsf{G}, \tau)\right\}$.*

▶ **Theorem 5.10.** *The divisibility topology is Noetherian.*

Proof. We prove that $\mathsf{E}_\Diamond$ is a topology expander and conclude thanks to Theorem 3.21.

1. Let us prove that $\mathsf{E}_\Diamond$ sends Noetherian topologies to Noetherian topologies. This is because it is the upwards closure of the image of a Noetherian topology through δ.
2. Let us show that $\mathsf{E}_\Diamond$ is monotone.
   Consider $\tau \subseteq \tau'$ two topologies on $\mu\mathsf{G}$. Let us write $X := (\mu\mathsf{G}, \tau)$ and $Y := (\mu\mathsf{G}, \tau')$. By definition of the inclusion of topologies, there exists an embedding $\iota \colon X \to Y$ in Top whose underlying function is the identity on $\mu\mathsf{G}$. Because G' preserves embeddings, G'_ι is an embedding from $\mathsf{G}'(X)$ to $\mathsf{G}'(Y)$, that is, an embedding from $(\mathsf{G}'(\mu\mathsf{G}), \mathsf{G}'(\tau))$ to $(\mathsf{G}'(\mu\mathsf{G}), \mathsf{G}'(\tau'))$. Moreover, $U\mathsf{G}'_\iota = \mathsf{G}_{U\iota} = \mathsf{G}_{\mathrm{Id}_{\mu\mathsf{G}}} = \mathrm{Id}_{\mu\mathsf{G}}$. As a consequence, $\mathsf{G}'(\tau) \subseteq \mathsf{G}'(\tau')$ and $\mathsf{E}_\Diamond(\tau) \subseteq \mathsf{E}_\Diamond(\tau')$.
3. Let us consider a Noetherian topology τ such that $\tau \subseteq \mathsf{E}_\Diamond(\tau)$, H closed in τ, and prove that $\mathsf{E}_\Diamond(\tau)|H \subseteq \mathsf{E}_\Diamond(\tau|H)|H$. Because G is an analytic functor, we can assume without loss of generality that $\mathsf{G}(H) \subseteq \mathsf{G}(\mu\mathsf{G})$.

   ▷ Claim 5.11. $\delta^{-1}(H) \subseteq G(H)$

   Proof. Let $t \in H$, because H is downwards closed for $\sqsubseteq$, for every $u \in \mathrm{supp}(\delta^{-1}(t))$, $u \in H$. As a consequence, $\mathrm{supp}(\delta^{-1}(t)) \subseteq H$, and this means that $\delta^{-1}(t) \in \mathsf{G}(H)$. ◁

   Let $U = \uparrow_{\sqsubseteq}\delta(V)$ be an open set of $\mathsf{E}_\Diamond(\tau)$. Notice that H is a closed subset of $\mathsf{E}_\Diamond(\tau)$ because $\tau \subseteq \mathsf{E}_\Diamond(\tau)$. Therefore,

$$\begin{aligned} U \cap H &= (\uparrow_{\sqsubseteq}\delta(V)) \cap H = \uparrow_{\sqsubseteq}(\delta(V) \cap H) \cap H = \uparrow_{\sqsubseteq}(\delta(V) \cap \delta(\mathsf{G}(H))) \cap H \\ &= \uparrow_{\sqsubseteq}\delta(V \cap \mathsf{G}(H)) \cap H \end{aligned}$$

To conclude that $U \cap H$ is open in $\mathsf{E}_{\Diamond}(\tau|H)|H$ it suffices to show that $V \cap \mathsf{G}(H)$ can be rewritten as $W \cap \mathsf{G}(H)$ where W is open in $\mathsf{G}'(\mu\mathsf{G}, \tau|H)$. Let us consider two maps $e_1 : (H, \tau_H) \to (\mu\mathsf{G}, \tau)$, and $e_2 : (H, \tau_H) \to (\mu\mathsf{G}, \tau|H)$. These two maps are embeddings, hence preserved by G'. As a consequence, $V \cap \mathsf{G}(H) = (\mathsf{G}'_{e_1})^{-1}(V)$, which is open. Because G'_{e_2} is an embedding, there exists a W open in $\mathsf{G}'(\mu\mathsf{G}, \tau|H)$ such that $(\mathsf{G}'_{e_2})^{-1}(W) = V \cap \mathsf{G}(H)$. This can be rewritten, $W \cap \mathsf{G}(H) = V \cup \mathsf{G}(H)$. ◀

As a sanity check, we can apply Theorem 5.10 to the sets of finite words and finite trees, and recover the subword topology and the tree topology that were obtained in an ad-hoc fashion in Section 4. In addition to validating the usefulness of Theorem 5.10, we believe that these are strong indicators that the topologies introduced prior to this work were the right generalisations of Higman's word embedding and Kruskal's tree embedding in a topological setting, and addresses the canonicity issue of the aforementioned topologies.

▶ **Lemma 5.12.** *The subword topology over Σ^*, is the divisibility topology associated to the analytic functor $X \mapsto \mathbf{1} + \Sigma \times X$.*

Proof. It suffices to remark that the functions $\mathsf{E}_{\Diamond}$ and $\mathsf{E}_{\text{words}}$ have the same least fixed point, and conclude using Lemma 3.10. ◀

▶ **Lemma 5.13.** *The tree topology over $\mathsf{T}(\Sigma)$, is the divisibility topology associated to the analytic functor $X \mapsto X \times \Sigma^*$.*

Proof. It suffices to remark that the functions $\mathsf{E}_{\Diamond}$ and E_{tree} have the same least fixed point, and conclude using Lemma 4.3. ◀

5.2 Divisibility Preorders

We are now going to prove that the divisibility topology correctly generalises the corresponding notions on quasi-orderings. In the case of finite words, this translates to the equation $\mathsf{alex}(\leq)^* = \mathsf{alex}(\leq^*)$ [12, Exercise 9.7.30]. We will proceed to generalise this result to every divisibility topology by relating it to the divisibility preorder introduced by Hasegawa [16, Definition 2.7].

Given an analytic functor G and its lift G^O to quasi-orderings respecting embeddings and wqos, let us build a family A_i of quasi-orders and $e_i : A_i \to A_{i+1}$ of embeddings as follows:

- $A_0 = \emptyset$, $A_1 = \mathsf{G}^O(A_0)$ and e_0 is the empty map.
- $e_{n+1} = \mathsf{G}^O_{e_n}$ and A_{n+1} has as carrier set $\mathsf{G}(A_n)$ and preordering the transitive closure of the union of the two following relations: The one is the quasi-order $\mathsf{G}^O(A_n)$, and the other is the collection of $b \triangleleft a$ for each weak normal form $(X, z) \to^f (A_n, a)$ in $\mathrm{el}(\mathsf{G})$ and each b in the image of $X \to^f A_n \to^{e_n} A_{n+1}$.

The *divisibility ordering* $\preceq$ is the ω-inductive limit in the category Ord of the diagram $A_0 \to^{e_0} A_1 \to^{e_1} \cdots$. As remarked by Hasegawa, the maps e_n are injective order embeddings, and so are the morphisms $c_n : A_n \to \mu\mathsf{G}$ of the colimiting cone [16, Lemma 2.8]. Without loss of generality, we can assume that $A_0 \subseteq A_1 \dots$ and that the colimit $\mu\mathsf{G}$ is the union of the sets A_i for $0 \leq i < \omega$. In particular, the map δ is the identity map in this setting.

▶ **Lemma 5.14.** *$a \triangleleft b$ in A_{n+1} if and only if $a \in \mathrm{supp}(\delta^{-1}(b))$.*

Proof. Assume that $a \triangleleft b$, then $b \in \mathsf{G}(A_n)$ and there exists a weak normal form $(X, z) \to^f (A_n, b)$ such that $a \in f(X)$. As $(A_n, b) \to^\iota (\mu\mathsf{G}, b)$, $(X, z) \to^{f\iota} (\mu\mathsf{G}, b)$ is also a weak normal form [16, Lemma 1.5]. As a consequence, $a \in \iota(f(X))$ and $a \in \mathrm{supp}(\delta^{-1}(b))$.

Assume that $a \sqsubset b$, there exists a weak normal form $(X, z) \to^f (\mu\mathsf{G}, b)$ such that $a \in f(X)$. As $b \in \mathsf{G}(A_n)$ for some $n \in \mathbb{N}$, this means that $(A_n, b) \to^\iota (\mu\mathsf{G}, b)$ is an element of the slice category, hence that there exists g such that $(X, z) \to^g (A_n, b)$ is a weak normal form of b and $\iota \circ g = f$. In particular, $a \in g(X)$, hence $a \in \mathrm{supp}(\delta^{-1}(b))$. ◀

A direct consequence is that our substructure relation captures the height of the sets A_n in the following sense:

▷ Claim 5.15. If $a \sqsubset b$ and $b \in A_{n+1}$ then $a \in A_n$.

▷ Claim 5.16. For all $n \in \mathbb{N}$, A_n is a downwards closed subset of A_{n+1}.

Now, it is an easy check that the divisibility preorder on $\mu\mathsf{G}$ is compatible with substructures as this is true for the sets A_n.

▶ **Lemma 5.17.** *We have* $(\preceq\sqsubseteq)^* =\preceq$.

Proof. By induction we prove it on A_n using the fact that $a \sqsubset b$ and $b \in A_{n+1}$ implies $a \in A_n$, thanks to **??** 5.15**??** 5.16. ◀

▶ **Corollary 5.18.** *The Alexandroff topology of the divisibility preorder contains the divisibility topology.*

Proof. It suffices to prove that $\mathsf{E}_\Diamond(\mathsf{alex}(\preceq)) \subseteq \mathsf{alex}(\preceq)$. Let us consider an open set V of $\mathsf{E}_\Diamond(\mathsf{alex}(\preceq))$ of the form $\uparrow_\sqsubseteq \delta(U)$, where U is open in $\mathsf{alex}(\preceq)$. In particular, $U =\uparrow_\preceq U$. Notice that $\uparrow_\sqsubseteq\uparrow_\preceq U = U$ because of Lemma 5.17. We have proven that $V \in \mathsf{alex}(\preceq)$. ◀

▶ **Lemma 5.19.** *For all $n \in \mathbb{N}$,*

$$(\sqsubseteq\leq_{\mathsf{G}^O(A_n)})^* \sqsubset \quad = \quad \leq_{\mathsf{G}^O(A_n)}\sqsubset$$

Note that this equality is only over elements of A_{n+1}.

Proof. Let $n \in \mathbb{N}$. Only one inclusion is non trivial. We know that $\leq_{A_{n+1}}= (\leq_{\mathsf{G}^O(A_n)}\sqsubseteq)^*$. As the maps e_n is an order embedding, for every $a, b \in A_n$, $a \leq_{A_{n+1}} b$ implies $a \leq_{A_n} b$. In particular, $\leq_{A_{n+1}}\sqsubset=\leq_{A_n}\sqsubset$. As e_n is monotone from A_n to $\mathsf{G}^O(A_n)$, $x \leq_{A_n} y$ implies $x \leq_{\mathsf{G}^O(A_n)} y$ and therefore $(\leq_{A_{n+1}}\sqsubset) \subseteq (\leq_{\mathsf{G}^O(A_n)}\sqsubset)$. ◀

▶ **Corollary 5.20.** *For all $n \in \mathbb{N}$,* $\leq_{\mathsf{G}^O(A_n)}\sqsubseteq=\leq_{A_{n+1}}$

▶ **Lemma 5.21.** *For all $n \in \mathbb{N}$,* $\mathsf{alex}(\preceq_n) \subseteq \mathsf{E}_\Diamond(\mathsf{alex}(\preceq_n))$, *where* $\preceq_n= \preceq|A_n$.

Proof. Let $x \in \mu\mathsf{G}$ and consider $U =\uparrow_{\preceq_n} x$, which is open in $\mathsf{alex}(\preceq_n)$. Let us write $V =\uparrow_{\preceq_{n+1}} \{y \colon x \preceq_n y\}$. It is clear that $U = V$, let us now prove that V is open in $\mathsf{E}_\Diamond(\mathsf{alex}(\preceq_n))$.

Thanks to Corollary 5.20, $V =\uparrow_\sqsubseteq\uparrow_{F(\preceq_n)} \{y \colon x \preceq_n y\}$. Moreover, $\uparrow_{\mathsf{G}^O(\preceq_n)} \{y \colon x \preceq_n y\}$ is open in $\mathsf{G}'(\mu\mathsf{G}, \mathsf{alex}(\preceq_n))$. As a consequence, we have proven that V is open in $\mathsf{E}_\Diamond(\mathsf{alex}(\preceq_n))$. ◀

▶ **Corollary 5.22.** $\mathsf{alex}(\preceq)$ *is contained in the divisibility topology.*

We are now ready to state our correctness theorem, i.e., that the divisibility topology is a correct generalisation to the topological setting of the divisibility preorder from Hasegawa.

▶ **Theorem 5.23.** *Let G' the be the lift of an analytic functor respecting Alexandroff topologies, Noetherian spaces, and embeddings. Then, the divisibility topology of $\mu\mathsf{G}$ is the Alexandroff topology of the divisibility preorder of $\mu\mathsf{G}$, which is a well-quasi-ordering.*

6 Concluding Remarks

We have provided a systematic way to place a Noetherian topology over an inductively defined datatype, which is correct with respect to its wqo counterpart whenever it exists. As a byproduct, we obtained a uniform framework that simplifies existing proofs, and serves as an indicator that the pre-existing topologies were the "right generalisations" of their quasi-order counterparts. Let us now briefly highlight some interesting properties of the underlying theory.

Differences with the existing categorical frameworks. The existing categorical frameworks are built around specific kind of functors [16, 8], while the notion of topology expander only requires talking about one specific set. This allows proving that the ordinal subword topology and the α-branching trees are Noetherian, while these escape both the realm of wqos, and of "well-behaved functors" having finite support functions.

Quasi-analytic functors. In fact, the proof of Theorem 5.10, never relies on the finiteness of the support of an element. This means that the definition of analytic functors can be loosened to allow non finite weak normal forms. We do not know whether this notion of "quasi-analytic functor" already exists in the literature.

Transfinite iterations. As the reader might have noticed, all of the least fixed points considered in this paper are obtained using at most ω steps. This is because the topology expanders that are presented in the paper are all Scott-continuous, i.e., they satisfy the equation $\mathsf{E}(\sup_i \tau_i) = \sup_i \mathsf{E}(\tau_i)$. While Theorem 3.21 does apply to non Scott-continuous topology expanders, we do not know any reasonable example of such expander.

Lack of ordinal invariants. Even though our proof that the ordinal subword topology is Noetherian is shorter than the original one, it actually provide less information. In particular, it does not provide a bound for ordinal rank of the lattice of closed sets (called the *stature* of $\Sigma^{<\alpha}$), whereas a clear bound is provided by the previous approach Goubault-Larrecq et al. [15, Proposition 33]. This limitation already appears in the existing categorical frameworks [16, 8], and we believe that this is inherent to the use of minimal bad sequence arguments.

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