Preservation Theorems Through the Lens of Topology

A toolset for studying preservation theorems

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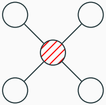
Preservation Theorems

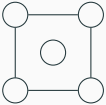
Illustrating the notions

A sentence over graphs

$$\varphi \triangleq \forall x. \exists y. \neg(xEy) \land x \neq y$$



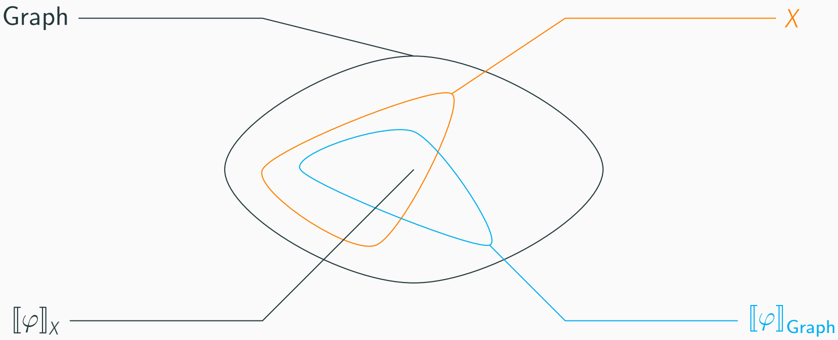
✗



✓

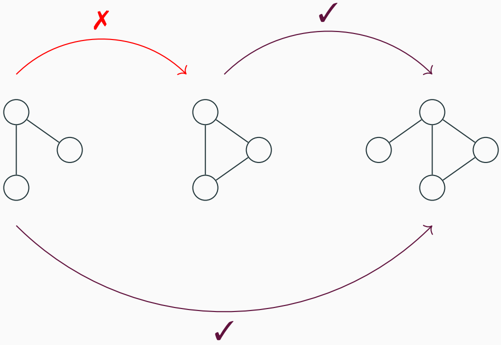
Models of a sentence

$$[\![\varphi]\!] : \text{Graph} \to \{0,1\} \qquad [\![\varphi]\!]_X \triangleq \{G \in X \mid G \models \varphi\}$$



A natural order for structures

$$A \subseteq_i B \iff \exists h\colon A \hookrightarrow B, h(E_A) = E_B$$



A natural order for truth values

$$0 \leq 1$$

Special kinds of sentences

$$[\![\varphi]\!] : (\mathsf{Graph}, \subseteq_i) \to (\{0,1\}, \leq)$$

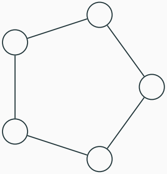
Monotone sentences over X

- $([\![\varphi]\!])_{|X}$ is *non-decreasing*
- $[\![\varphi]\!]_X = \uparrow [\![\varphi]\!]_X \triangleq \{G \in X \mid \exists H \in [\![\varphi]\!]_X , H \subseteq_i G\}$
- For all $(G_1, G_2) \in X^2$ such that $G_1 \subseteq_i G_2$ and $G_1 \models \varphi$, $G_2 \models \varphi$.

Preservation Theorems

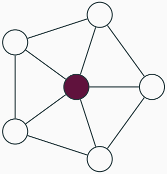
Evaluating φ

Recall that $\varphi \triangleq \forall x. \exists y. \neg(xEy) \land x \neq y$.



$\models \varphi$ ✓

$\subseteq_i$



$\models \varphi$ ✗

Not Monotone!

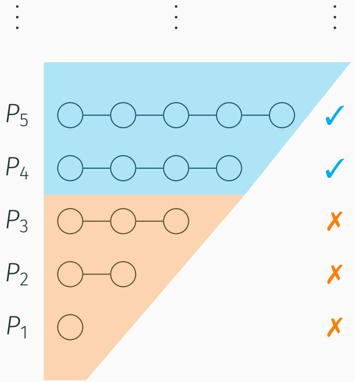
Structure of paths

- Totally ordered for $\subseteq_i$
- The sentence φ is monotone

Rewriting φ

$$[\![\varphi]\!]_{\mathcal{P}} = \uparrow\{P_4\}$$

$$\begin{aligned}\varphi \equiv_{\mathcal{P}} & \exists x_1, x_2, x_3, x_4. \\ & x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \\ & x_1 \neq x_4 \wedge x_2 \neq x_3 \wedge \\ & x_2 \neq x_4 \wedge x_3 \neq x_4 \\ \triangleq\ & \psi_4\end{aligned}$$

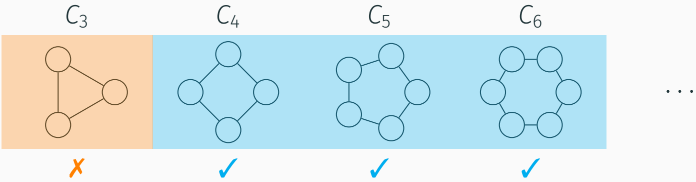


Structure of cycles

- Infinite *antichain*
- The sentence φ is monotone

Rewriting φ

- $[\![\varphi]\!]_{\mathcal{C}} = \uparrow\{P_4\} \cup \{C_4\}$
- $\varphi \equiv_{\mathcal{C}} \psi_4$



P_4 

Recap

We considered φ and studied whether it was *monotone*, *equivalent to an existential sentence* and *had finitely many minimal models* over different classes.

Space	Monotone	Existential	Compact[1]
Graph	✗	✗	✗
$\mathcal{C}$	✓	✓	✗
$\mathcal{P}$	✓	✓	✓

[1] Has finitely many minimal models

Preservation Theorems

Łoś-Tarski and beyond

Łoś-Tarski's preservation theorem over $X \subseteq \mathsf{Struct}(\sigma)$

Connecting an order to a syntactic fragment.

$$\forall \sigma . \forall \varphi \in \mathsf{FO}[\sigma].$$

$$[\![\varphi]\!]_X = \uparrow [\![\varphi]\!]_X \iff \exists \psi \in \mathsf{EFO}[\sigma], [\![\varphi]\!]_X = [\![\psi]\!]_X$$

Struct(σ)	Order	Fragment	FinStruct(σ)
Łós-Tarski ✓	$\subseteq_i$	EFO	✗ [Tait, 1959]
Tarski-Lyndon ✓	$\subseteq$	PFO	✗ [Ajtai and Gurevich, 1994]
H.P.T. ✓	$\to$	EPFO	✓ [Rossman, 2008]

Consequence

No relativisation result in general.

Preservation Theorems

Logically defined pre-spectral spaces

Pre-order	Topology
$\leq$	τ
upwards-closed	open
monotone	continuous
$\uparrow F$, F finite	compact

Converting

- A topology τ has a specialisation pre-order, $x \leq_\tau y \triangleq \forall U \in \tau, x \in U \implies y \in U$.
- A pre-order $\leq$ has *several* topologies. We will use Alexandroff $\tau_\leq \triangleq \{\uparrow U \mid U \subseteq X\}$.

The case of cycles

- $\tau_{\subseteq_i}$ is the discrete topology.
- EFO defines $\subseteq_i$.

Topological Solution

$$\tau_{\mathcal{C}} \triangleq \{F \cup \uparrow\{P_n\} \mid F \subseteq_f \mathcal{C} \wedge n \geq 1\} \cup \{\emptyset\}$$

Note: The specialisation preorder of $\tau_{\mathcal{C}}$ is $\subseteq_i$.

Topological Solution

$$\tau_{\mathcal{C}} \triangleq \{F \cup \uparrow\{P_n\} \mid F \subseteq_f \mathcal{C} \land n \geq 1\} \cup \{\emptyset\}$$

The case of cycles

- Every open set is compact
- Every compact-open set is definable in EFO
- Every continuous sentence $\varphi \in \mathrm{FO}[\sigma]$ is definable in EFO.

Some perspectives

- Some sentences in EFO are *not* continuous in $(\mathcal{C}, \tau_{\mathcal{C}})$!
- But if φ is not continuous, $\neg\varphi$ is.
- The complementary of a compact-open set is definable in EFO
- Every sentence $\varphi \in \mathsf{FO}[\sigma]$ is definable in EFO.

Why topology matters

Slight variations

- $\downarrow\mathcal{C}$ does not validate Łoś-Tarski's preservation theorem (adding points make things worse).
- Graphs of degree ≤ 2 validates Łoś-Tarski's preservation theorem [Atserias et al., 2008].

Spaces of interest

- $X \subseteq \mathsf{FinStruct}(\sigma)$ the support
- τ a topology over X
- $\mathcal{K}^\circ(X)$ the compact-open sets
- $\mathrm{FO}[\sigma]$ the logic

Motto

Definable open sets are compact.

Logically-presented pre-spectral spaces (lpps)

For a triplet $\langle X, \tau, \mathsf{FO}[\sigma] \rangle$

Pre-spectral $\tau = \langle \mathcal{K}^\circ(X) \rangle$ and $\mathcal{K}^\circ(X)$ is a lattice (see [Dickmann et al., 2019])

Presented $\tau \cap [\![\mathsf{FO}]\!]_X \subseteq \mathcal{K}^\circ(X)$

Remark

In a lpps, $\tau \cap [\![\mathsf{FO}]\!]_X = \mathcal{K}^\circ(X)$

Equivalence with preservation theorems

Let F be a fragment of $\mathsf{FO}[\sigma]$ and $\leq$ a pre-order on $X \subseteq \mathsf{FinStruct}(\sigma)$ generated by F. Assume that $\leq$ is downwards closed in $\mathsf{FinStruct}(\sigma)$.

There is a preservation theorem for $(X, \leq, \mathsf{F})$ **iff** $\langle X, \tau_{\leq}, \mathsf{FO}[\sigma] \rangle$ is a lpps.

Note

If $\langle X, \tau_{\leq}, \mathsf{FO}[\sigma] \rangle$ is a lpps, then $(X, \leq, \mathsf{F})$ always satisfies a preservation theorem.

A practical toolset

Combining previous examples

A simple union

Consider $\mathcal{U} \triangleq \mathcal{C} \cup \mathcal{P}$.

- Does $\mathcal{U}$ satisfy Łoś-Tarski's preservation theorem?
- Is $\langle \mathcal{U}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$ a lpps?
- Can we adapt $\tau_{\mathcal{C}}$ to $\mathcal{U}$?

A simple union

Consider $\mathcal{U} \triangleq \mathcal{C} \cup \mathcal{P}$.

- Does $\mathcal{U}$ satisfy Łoś-Tarski's preservation theorem?
  ✗ $\forall x. \mathsf{deg}(x) = 2$
- Is $\langle \mathcal{U}, \tau_{\subseteq_i}, \mathrm{FO}[E] \rangle$ a lpps?
- Can we adapt $\tau_{\mathcal{C}}$ to $\mathcal{U}$?

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Consider $\mathcal{U} \triangleq \mathcal{C} \cup \mathcal{P}$.

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  ✗ $\forall x. \operatorname{deg}(x) = 2$
- Can we adapt $\tau_{\mathcal{C}}$ to $\mathcal{U}$?
  ✓

A simple union

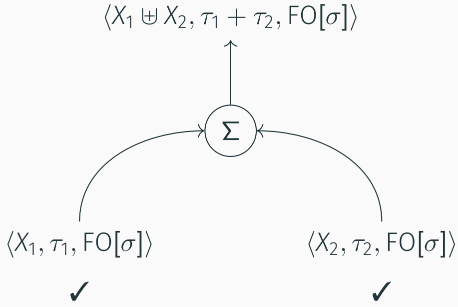
$\langle X_1, \tau_1, \mathrm{FO}[\sigma] \rangle$ $\langle X_2, \tau_2, \mathrm{FO}[\sigma] \rangle$

A simple union

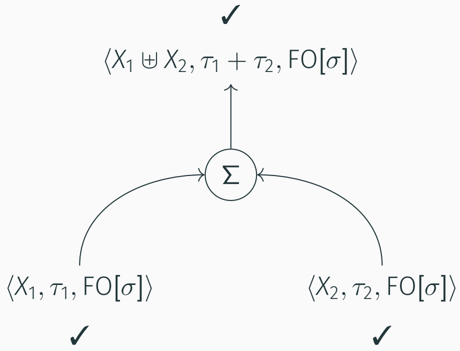
$\langle X_1, \tau_1, \mathsf{FO}[\sigma] \rangle$ ✓

$\langle X_2, \tau_2, \mathsf{FO}[\sigma] \rangle$ ✓

A simple union



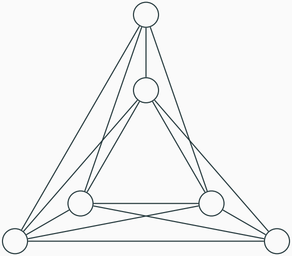
A simple union



A practical toolset

More complex construction

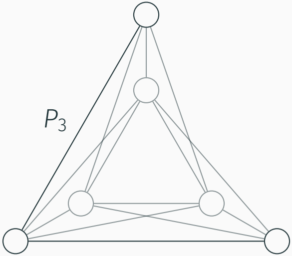
Consider $\mathcal{B} \triangleq \mathcal{C} \bowtie \mathcal{P} \triangleq \{C_i \bowtie P_j \mid (C_i, P_j) \in \mathcal{C} \times \mathcal{P}\}$.



$C_3 \bowtie P_3$

A complex class of graphs

Consider $\mathcal{B} \triangleq \mathcal{C} \bowtie \mathcal{P} \triangleq \{C_i \bowtie P_j \mid (C_i, P_j) \in \mathcal{C} \times \mathcal{P}\}$.



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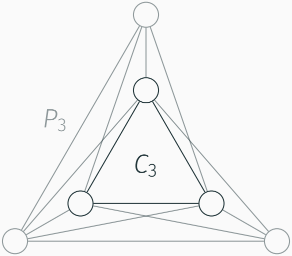

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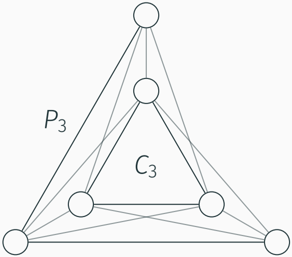


$C_3 \bowtie P_3$

Fiddling around

Support	Topology	Łoś-Tarski	Lpps
$\mathcal{P}$	$\tau_{\subseteq_i}$	✓	✓
$\mathcal{C}$	$\tau_{\subseteq_i}$	✓	✗
$\mathcal{C}$	$\tau_{\mathcal{C}}$	✓	✓
$\mathcal{C} \bowtie \mathcal{P}$	$\tau_{\subseteq_i}$	?	?
$\mathcal{C} \bowtie \mathcal{P}$	?	?	?

A practical toolset

Stability properties

Negative result

Assume $\langle \mathcal{C} \bowtie \mathcal{P}, \tau_{\subseteq_i}, \mathrm{FO}[E] \rangle$ *is a lpps.*

Restrictions to definable open or definable closed sets

$$\langle X, \tau, \mathrm{FO}[\sigma] \rangle \text{ lpps} \wedge [\![\varphi]\!]_X \in \tau \implies \langle [\![\varphi]\!]_X, \tau, \mathrm{FO}[\sigma] \rangle \text{ lpps}$$

Continuous FO-interpretation

$$\langle X, \tau, \mathrm{FO}[\sigma] \rangle \text{ lpps} \wedge f\colon (X, \tau) \twoheadrightarrow (Y, \theta) \implies \langle Y, \theta, \mathrm{FO}[\sigma] \rangle \text{ lpps}$$

Negative result

Assume $\langle \mathcal{C} \bowtie \mathcal{P}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$ *is a lpps.*

Restrictions to definable open or definable closed sets

- $\psi_1 \triangleq \exists x.\, \mathsf{deg}(x) \geq 4.$
- $\psi_2 \triangleq \forall x_1, x_2.\, \mathsf{deg}(x_1) \geq 4 \land \mathsf{deg}(x_2) \geq 4 \implies x_1 = x_2.$

Continuous FO-interpretation

$$\langle X, \tau, \mathsf{FO}[\sigma] \rangle \text{ lpps} \land f\colon (X, \tau) \twoheadrightarrow (Y, \theta) \implies \langle Y, \theta, \mathsf{FO}[\sigma] \rangle \text{ lpps}$$

Assume $\langle \mathcal{C} \bowtie \mathcal{P}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$ *is a lpps.*

Restrictions to definable open or definable closed sets

$$\langle \mathcal{C}_{\geq 4} \bowtie \{P_1\}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle \text{ is a lpps}$$

Continuous FO**-interpretation**

$$\langle X, \tau, \mathsf{FO}[\sigma] \rangle \text{ lpps} \wedge f \colon (X, \tau) \twoheadrightarrow (Y, \theta) \implies \langle Y, \theta, \mathsf{FO}[\sigma] \rangle \text{ lpps}$$

Negative result

Assume $\langle \mathcal{C} \bowtie \mathcal{P}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$ *is a lpps.*

Restrictions to definable open or definable closed sets

$$\langle \mathcal{C}_{\geq 4} \bowtie \{P_1\}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle \text{ is a lpps}$$

Continuous FO-interpretation

- $\delta(x) \triangleq \mathrm{deg}(x) \leq 3$.
- $\psi_E(x, y) \triangleq E(x, y)$.

Negative result

Assume $\langle \mathcal{C} \bowtie \mathcal{P}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$ *is a lpps.*

Restrictions to definable open or definable closed sets

$$\langle \mathcal{C}_{\geq 4} \bowtie \{P_1\}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle \text{ is a lpps}$$

Continuous FO-interpretation

$$\langle \mathcal{C}_{\geq 4}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle \text{ is a lpps}$$

Absurd.

Fiddling around

Support	Topology	Łoś-Tarski	Lpps
$\mathcal{P}$	$\tau_{\subseteq_i}$	✓	✓
$\mathcal{C}$	$\tau_{\subseteq_i}$	✓	✗
$\mathcal{C}$	$\tau_{\mathcal{C}}$	✓	✓
$\mathcal{C} \bowtie \mathcal{P}$	$\tau_{\subseteq_i}$	?	✗
$\mathcal{C} \bowtie \mathcal{P}$	?	?	?

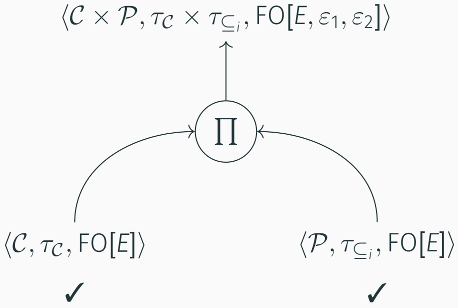
Positive result

$\langle \mathcal{C}, \tau_{\mathcal{C}}, \mathsf{FO}[E] \rangle$ $\langle \mathcal{P}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$

Positive result

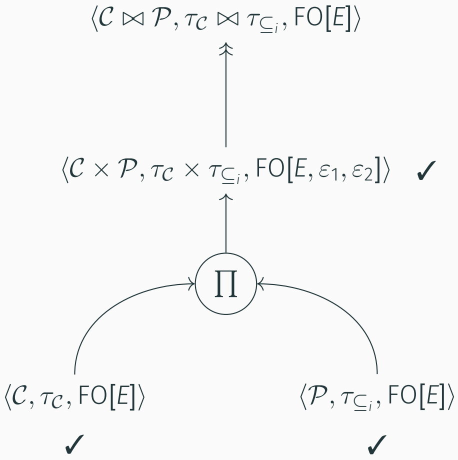
$\langle \mathcal{C}, \tau_{\mathcal{C}}, \mathsf{FO}[E] \rangle$ ✓

$\langle \mathcal{P}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$ ✓



$\langle \mathcal{C} \times \mathcal{P}, \tau_{\mathcal{C}} \times \tau_{\subseteq_i}, \mathsf{FO}[E, \varepsilon_1, \varepsilon_2] \rangle$
$\prod$
$\langle \mathcal{C}, \tau_{\mathcal{C}}, \mathsf{FO}[E] \rangle$
$\langle \mathcal{P}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$
✓
✓

Positive result



Positive result

$$\langle \mathcal{C} \bowtie \mathcal{P}, \tau_{\mathcal{C}} \bowtie \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle \quad ✓$$

$$\uparrow$$

$$\langle \mathcal{C} \times \mathcal{P}, \tau_{\mathcal{C}} \times \tau_{\subseteq_i}, \mathsf{FO}[E, \varepsilon_1, \varepsilon_2] \rangle \quad ✓$$

$$\uparrow$$

$$\prod$$

$$\langle \mathcal{C}, \tau_{\mathcal{C}}, \mathsf{FO}[E] \rangle \qquad \langle \mathcal{P}, \tau_{\subseteq_i}, \mathsf{FO}[E] \rangle$$

✓ ✓

Fiddling around

Support	Topology	Łoś-Tarski	Lpps
$\mathcal{P}$	$\tau_{\subseteq_i}$	✓	✓
$\mathcal{C}$	$\tau_{\subseteq_i}$	✓	✗
$\mathcal{C}$	$\tau_{\mathcal{C}}$	✓	✓
$\mathcal{C} \bowtie \mathcal{P}$	$\tau_{\subseteq_i}$	?	✗
$\mathcal{C} \bowtie \mathcal{P}$	$\tau_{\mathcal{C}} \bowtie \tau_{\subseteq_i}$	?	✓

Consider $(\mathcal{C} \bowtie \mathcal{P}, \tau_{\mathcal{C}} \bowtie \tau_{\subseteq_i})$.

- Every compact-open set of is definable in EFO.
- Every definable open set is definable in EFO.

Warning!

Some EFO sentences are not continuous.

Exercise

If φ is monotone for $\subseteq_i$, and not continuous, then φ has finitely many models, hence is definable in EFO.

Fiddling around

Support	Topology	Łoś-Tarski	Lpps
$\mathcal{P}$	$\tau_{\subseteq_i}$	✓	✓
$\mathcal{C}$	$\tau_{\subseteq_i}$	✓	✗
$\mathcal{C}$	$\tau_{\mathcal{C}}$	✓	✓
$\mathcal{C} \bowtie \mathcal{P}$	$\tau_{\subseteq_i}$	✓	✗
$\mathcal{C} \bowtie \mathcal{P}$	$\tau_{\mathcal{C}} \bowtie \tau_{\subseteq_i}$	✓	✓

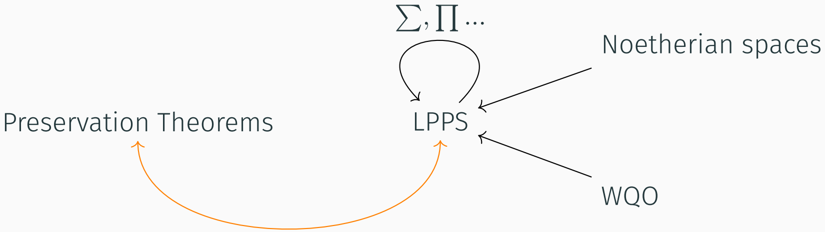
A practical toolset

Details of the toolset

Closure properties

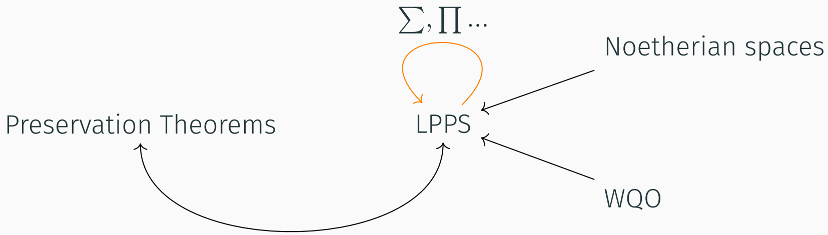
Constructor	Restrictions
Subset	definable, open or closed
Image of a morphism	
Sum	finite
Product	finite
Density	
Projective limit	$\star$

Brief recap

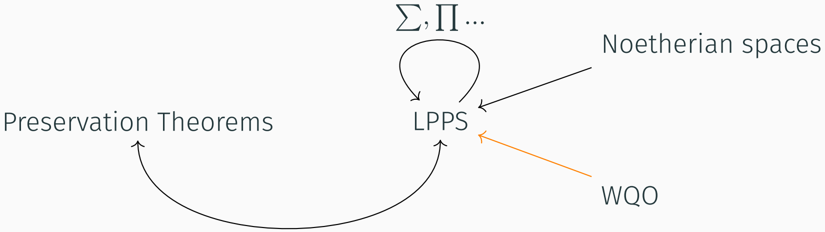


Preservation to LPPS (Theorem 3.4)

The Approach

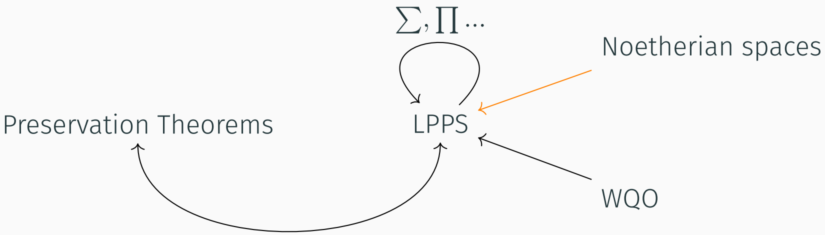


Closure Properties (Propositions 5.4, 5.5, 5.8)

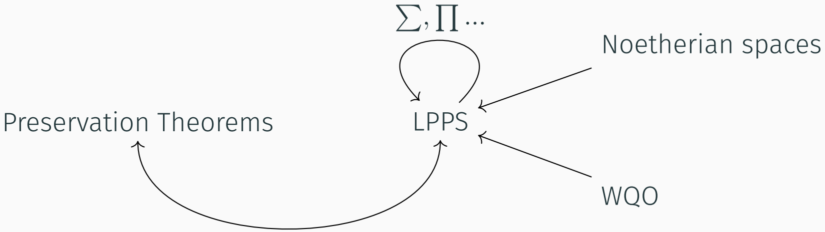


Using known spaces (well-quasi-orderings)

The Approach



Using known spaces (noetherian spaces)


$\sum, \prod \dots$
Noetherian spaces
Preservation Theorems
LPPS
WQO

Thank You!

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Journal of the ACM, 55(3):15:1–15:53.

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A counterexample to a conjecture of Scott and Suppes.
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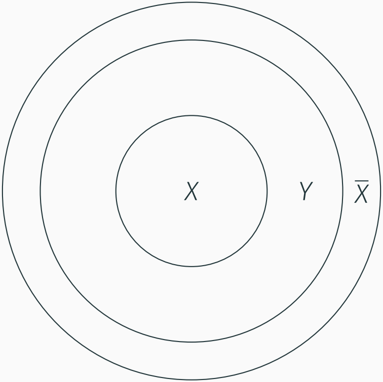
Rossman's Proof of the H.P.T.

$$X_n \triangleq \langle X, \tau_{\to_n}, \mathrm{FO}[\sigma] \rangle \qquad Y_n \triangleq \left\langle X, \tau_{\to}, \mathrm{FO}^n[\sigma] \right\rangle$$

1. The projective limit of Y_n in Top is $\langle X, \tau_{\to}, \mathrm{FO}[\sigma] \rangle$.
2. $\exists \rho. \forall n. \mathcal{K}^\circ(Y_n) \subseteq \mathcal{K}^\circ\left(X_{\rho(n)}\right)$
3. Hence $\langle X, \tau_{\to}, \mathrm{FO}[\sigma] \rangle$ is lpps.

Logical Closure



$$\forall \varphi \in \mathrm{FO}[\sigma], [\![\varphi]\!]_X \neq \emptyset \iff [\![\varphi]\!]_{\overline{X}} \neq \emptyset$$

What we are hoping for

1. New constructions (finite words, infinite words, trees, …).
2. Iterative constructions.
3. Handle smaller fragments $\exists^k\forall$.
4. Locally lpps is the same as lpps.