Basic Operational Preorders for Algebraic Effects

With a pinch of non-determinism and probabilities

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17 / 12 / 2021

Under the supervision of Sylvain Schmitz and Jean Goubault-Larrecq
A deceptively long introduction

Effects
Algebraic effects

PCF+Effects


Concrete implementations with handlers

- Haskell implementations (Fused Effects, Polysemy, Eff, ...)

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What’s in my bag?

Local state, global state, exceptions, non-determinism, random numbers, logging, input/output, continuations.

Many of the effects listed are in fact algebraic (modeled by a Lawvere theory) and therefore share nice properties.

Program equivalence and how to deal with it

- Dal Lago et al. (2017)
- Johann et al. (2010)
A deceptively long introduction

What does it look like concretely?
Simple programming language: Types

Non polymorphic term types

\[ \tau ::= \text{Nat} | \tau \to \tau \]

Polymorphic effect types

<table>
<thead>
<tr>
<th>Example</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lookup</td>
<td>( \sigma : \alpha^{\text{Nat}} \to \alpha )</td>
</tr>
<tr>
<td>Binary choice</td>
<td>( \sigma : \alpha^n \to \alpha )</td>
</tr>
<tr>
<td>Update</td>
<td>( \sigma : \text{Nat} \times \alpha^n \to \alpha )</td>
</tr>
<tr>
<td>Generic case</td>
<td>( \sigma : \text{Nat} \times \alpha^{\text{Nat}} \to \alpha )</td>
</tr>
</tbody>
</table>
What about polymorphism?

Adding polymorphism for terms

- Done in Johann et al. (2010), not the main technicality
- Very useful in practice Wadler (1989); Pitts (2000): “there are not much functions of type \( \forall \alpha. \alpha \to \alpha \).”
Terms, $\sigma \in \Sigma$

\[ M ::= x \mid \lambda x : \tau. M \mid MM \mid \text{fix } M \mid Z \mid SM \]
\[ \mid \text{case } M \text{ of } Z \Rightarrow M; S(x) \Rightarrow M \]
\[ \mid \sigma(M, \ldots, M) \]
\[ \mid \sigma(M; M, \ldots, M) \]

Values

\[ V ::= \lambda x : \tau. M \mid Z \mid SV \]
A deceptively long introduction

Reasoning about effects
Contextual equivalence

\[ M \equiv_{\text{ctx}} M' \]

\[ \forall C[-], \forall n, C[M] : \text{Nat}, C[M'] : \text{Nat}, C[M] \downarrow n \iff C[M'] \downarrow n \]

Issues

- Can capture free variables;
- More suitable for proving non-equivalence;
- Taylor made for termination.

Two amidst many alternatives

- Bisimilarity and bisimulations Dal Lago et al. (2017);
- Logical relations Johann et al. (2010).
In the presence of effects, function extentionality is not a good way to reason:

\[
\lambda x. \text{OR}(1, 2) \ ? \ \text{OR}(\lambda x.1, \lambda x.2)
\]
In the presence of effects, function extensionality is not a good way to reason:

\[ \lambda x.\text{OR}(1, 2) \not\equiv_{\text{ctx}} \text{OR}(\lambda x.1, \lambda x.2) \]
Why can’t we easily reason?

In the presence of effects, function extentionality is not a good way to reason:

\[ \lambda x.\text{OR}(1, 2) \not\equiv_{\text{ctx}} \text{OR}(\lambda x.1, \lambda x.2) \]

\[ C[\_\_] = (\lambda f. f\emptyset + \emptyset)[\_] \]
The zen way of building contextual preorders

The work of Johann et al. (2010)
Stacks, terms, trees

Reduce a pair \( \langle S, M \rangle \) to a tree \( |S, M| \) of effects where leaves are values.

Granted \( \precsim \) is a preorder over \( \text{Tree}_{\text{Nat}} \) the free continuous \( \Sigma \)-algebra over \( \text{Nat} \).

Contextual Preorder

The contextual preorder is the largest compatible (closed under context) and \( \precsim \)-adequate (included in \( \precsim \) at ground type) relation.
Example of trees

Let $\Sigma = \{\text{pr}\}$ be a signature containing one binary effect construction.
Generic operational meta-theory

**Input:** A preorder $\preceq$ for type Nat

**Output:** A logical relation (!) on programs that characterises contextual preorder (Morris-Style)
Some relations are more equal than others

† Admissible If $t_i \preceq t_i'$ and $(t_i)_i, (t'_i)_i$ are ascending chains then

\[
\bigsqcup_i t_i \preceq \bigsqcup_i t'_i
\]

‡ Compatible If $t_i \preceq t'_i$ and $\sigma \in \Sigma$ then $\sigma(t_1, \ldots) \preceq \sigma(t'_1, \ldots)$.

◊ Substitutive Given $\rho : \text{Tree}_{\text{Nat}} \to \text{Tree}_{\text{Nat}}$, if $t \preceq t'$ then $t\rho \preceq t'\rho$

□ Compositional Given $\rho, \rho' : \text{Tree}_{\text{Nat}} \to \text{Tree}_{\text{Nat}}$, if $t \preceq t'$ and $\rho \preceq \rho'$ then $t\rho \preceq t'\rho'$

† ∧ ‡ ∧ ◊ ⇐⇒ † ∧ □
### Examples, counter examples

<table>
<thead>
<tr>
<th>Effect</th>
<th>Admissible</th>
<th>Compatible</th>
<th>Substitutive</th>
<th>Compositional</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hoare</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Smyth</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Countable Smyth</td>
<td>✗</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Valuations</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Exceptions</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Mixed Pr/Nd</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
</tr>
</tbody>
</table>

**Idea**

Denotational semantics provides admissibility and compatibility for free...
Bad operational preorder (not substitutive)

\[ \forall H \subseteq \text{Nat}, \sup_s \mathbb{P}(t/s \in H) \leq \sup_s \mathbb{P}(t'/s \in H) \]

Hint: substitute using:

- \(1 \mapsto \frac{7}{8} \cdot 0 + \frac{1}{8} \cdot 1, \ 2 \mapsto 1, \ 3 \mapsto \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 1\)

and compute for \(H = \{1\} \ldots\)
The zen way of building contextual preorders

Nice, but what does this mean?
Given a continuous $\Sigma$-algebra $D$ and a morphism $[\cdot] : \mathbb{N}_{\perp} \to D$ one can build the preorder $\preceq_{\text{den}}$. 

$$t \preceq_{\text{den}} t' \iff [t] \leq_D [t'] \quad (1)$$
Given a continuous $\Sigma$-algebra $D$ and a morphism $\lbrack \cdot \rbrack : N_{\perp} \to D$ one can build the preorder $\preceq_{\text{den}}$.

\[
\begin{array}{c}
N \\
\downarrow i
\end{array}
\begin{array}{c}
j \\
\rightarrow
\end{array}
\begin{array}{c}
D \\
\downarrow [\cdot]
\end{array}
\begin{array}{c}
\downarrow \text{Tree}(N)
\end{array}
\]

\[t \preceq_{\text{den}} t' \iff \lbrack t \rbrack \leq_{D} \lbrack t' \rbrack \quad (1)\]

Properties of $\preceq_{\text{den}}$

1. Automatically admissible (continuity)
2. Automatically compatible ($\Sigma$-algebra)
3. Not always compositional! (+, $\times$ interpreted naturally)
Factorisation

The map \([\cdot] : \mathbb{N} \rightarrow D\) is said to have the factorisation property if, for every function \(f : \mathbb{N} \rightarrow D\), there exists a continuous homomorphism \(h_f : D \rightarrow D\) such that \(f = h_f \circ [\cdot]\).

\[
\begin{array}{ccc}
\mathbb{N} & \xrightarrow{[\cdot]} & D \\
& \overset{f}{\searrow} & \downarrow h_f \\
& & D
\end{array}
\]

Consequence

We then have \(\lbrack t\sigma \rbrack = h_\sigma(\lbrack t \rbrack)\) which is continuous in \(t\) with a fixed \(\sigma\).
Proof

\[ f = [\cdot] \circ \rho \]

\[ \begin{array}{c}
\mathbb{N} \\
\downarrow \iota \end{array} \xrightarrow{\rho} \begin{array}{c}
\text{Tree}_\mathbb{N} \\
\downarrow [\cdot] \end{array} \xrightarrow{[\cdot]} \begin{array}{c}
\mathbb{D} \\
\downarrow h_f \end{array} \]

\[ \begin{array}{c}
\text{Tree}_\mathbb{N} \\
\downarrow [\cdot] \end{array} \xrightarrow{[\cdot]} \begin{array}{c}
\mathbb{D} \\
\end{array} \]
If \((T, \eta, \mu)\) is a monad over continuous \(\Sigma\)-algebras, the map \(\eta: \text{Nat} \rightarrow T\text{Nat}\) satisfies the factorisation property with \(h_f \triangleq f^\dagger\).
Kegelspitze Keimel and Plotkin (2017)
Scott closed convex subsets of valuations over $X$ ordered by inclusion.

Previsions Goubault-Larrecq (2016)
$L(X)$ is the set of lower semicontinuous maps from $X$ to $\mathbb{R}$. We use $\mathcal{L}(\mathcal{L}(X))$ to represent previsions.
Concretely
The zen way of building contextual preorders

Axiomatics
Let $\text{Vars}$ be a set of countably many distinct variables

$$\left( \bigwedge_{i \in I} e_i \leq e'_i \right) \implies e \leq e',$$

An effect theory $T$ is a set of Horn clauses.

**Order associated to a theory**

There exists a smallest admissible and compositional preorder $\preceq_T$ satisfying $T$. 
Concretely...

Bot: \( \bot \leq x \)

Prob: \( x \preceq x, \ x \preceq y = y \preceq x, \ (x \preceq y) \preceq (z \preceq w) = (x \preceq z) \preceq (y \preceq w) \)

Appr: \( x \preceq y \leq y \implies x \leq y \)

Nondet: \( x \lor x = x, \ x \lor y = y \lor x, \ x \lor (y \lor z) = (x \lor y) \lor z \)

Ang: \( x \lor y \leq x \)

Dem: \( x \lor y \geq x \)

Dist: \( x \preceq (y \lor z) = (x \preceq y) \lor (x \preceq z) \)

Figure 1: Horn theory for mixed probability and non determinism
Universal approximation scheme

Let $\frac{2^0 - 1}{2^0} t = \bot$ and $\frac{2^{(n+1)} - 1}{2^{(n+1)}} t = t$ or $\frac{2^n - 1}{2^n} t$, extend with $\frac{2^\infty - 1}{2^\infty} t = \bigcup_n \frac{2^n - 1}{2^n} t$.

In a reasonable interpretation of trees $\frac{2^n - 1}{2^n} t \ll t$ and $\frac{2^\infty - 1}{2^\infty} t = t$.

Removing implications

For any effect theory containing the Bot and Prob axioms, an admissible model satisfies the Appr axiom if and only if it satisfies the equation $\frac{2^\infty - 1}{2^\infty} x = x$. 
A nice equality $\preceq^{\text{ax}} = \preceq^{\text{den}}$

The hard part $\preceq^{\text{den}} \subseteq \preceq^{\text{ax}}$

(i) Over trees without or: well-known since Heckmann (1994)
   - Finite case: normal form
   - Infinite case: approximation

(ii) Over trees with finitely many or
   - Use distributivity to have a finite hat of or nodes
   - $t' \equiv_{\text{ax,den}} t \lor k$, when $k = \lambda_1 t_1 + \cdots + \lambda_n t_n$
   - $\forall i, \exists k_i := \lambda_1 t'_i + \cdots + \lambda_n t'_n$, $t_i \preceq^{\text{den}} k_i$
   - $t \preceq^{\text{ax}} t' \lor k_1 \lor \cdots \lor k_n \equiv_{\text{ax,den}} t'$

(iii) Over arbitrary trees using $\frac{2^n - 1}{2^n} t$, admissibility and the way-below relation
What admissibility? Why the way-below relation?

\[
\begin{align*}
&\left\lfloor \frac{2^n - 1}{2^n} t \right\rfloor \ll [t] \leq \bigsqcup_m \left\lfloor \frac{2^m - 1}{2^m} t' \right\rfloor \\
&\text{Hence there exists an } m \text{ such that} \\
&\left\lfloor \frac{2^n - 1}{2^n} t \right\rfloor \leq \left\lfloor \frac{2^m - 1}{2^m} t' \right\rfloor \\
&\text{And conclude using the approximation.}
\end{align*}
\]
The zen way of building contextual preorders

Can we forget about domain theory?
Given a strategy $s : \{ l, r \}^* \to \{ l, r \}$ and a tree $t$ evaluate $t | s$.

$$t \preceq^{\text{op}} t' \iff \forall h : \mathbb{N} \to [0, \infty] \, \sup_s E_{t|s}(h) \leq \sup_s E_{t'|s}(h)$$

1. Compatible: easy
2. Substitutive: easy
3. Admissible: the function $G_h : (s, t) \mapsto E_{t|s}(h)$ is continuous and the set of strategies is compact.
The correspondence between the operational preorder and the denotational one is observed through the isomorphism between \( \mathcal{L}(\mathcal{L}(X)) \) and \( \mathcal{S}_1 \mathcal{V}_X \) noticed by Keimel and Plotkin (2017) and Goubault-Larrecq (2016)

\[
\Lambda : A \mapsto \left( f \mapsto \inf_{\mu \in A} \int_{n \in \mathbb{N}} f(n) d\mu \right)
\]
Missing

What I did not tell
Probabilities and non-determinism are still a thing

- Call-by-push-value and full abstraction for PCF with probabilities and non-determinism Goubault-Larrecq (2019)
- Weak distributive laws allow to combine “naturally” probabilities and non-determinism Goy and Petrisan (2020)
- And many more!
Thank you 😊


