

ATLaS: The Algorithmic Theory of Linear Systems

Joël Ouaknine

Max Planck Institute for Software Systems

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Classical Dynamical Systems Theory

LES MÉTHODES NOUVELLES

DE LA

MÉCANIQUE CÉLESTE

PAR

H. POINCARÉ,

MEMBRE DE L'INSTITUT,
PROFESSEUR À LA FACULTÉ DES SCIENCES.

TOME II.

Méthodes de MM. Newcomb, Gylden, Lindstedt et Bohlén.



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GAUTHIER-VILLARS ET FILS, IMPRIMEURS-LIBRAIRES

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

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- asymptotic/periodic analysis
- measure-theoretic/topological questions

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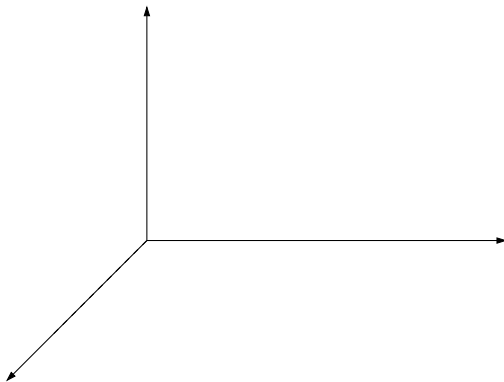
In general: very difficult to give exact answers to specific algorithmic questions about orbits

Example: a discrete linear dynamical system

Ambient space: \mathbb{R}^3

Starting point: $\mathbf{x} \in \mathbb{Q}^3$

Linear transformation: $\mathbf{M} \in \mathbb{Q}^{3 \times 3}$

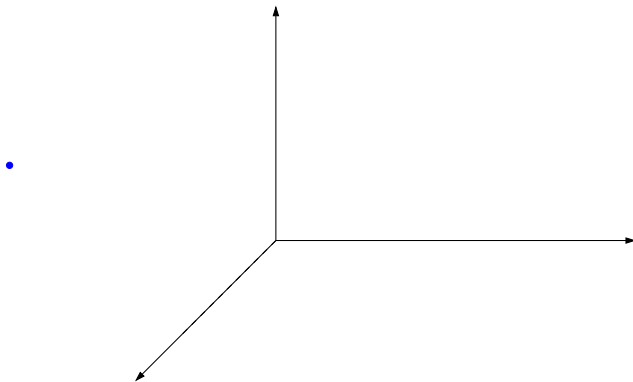


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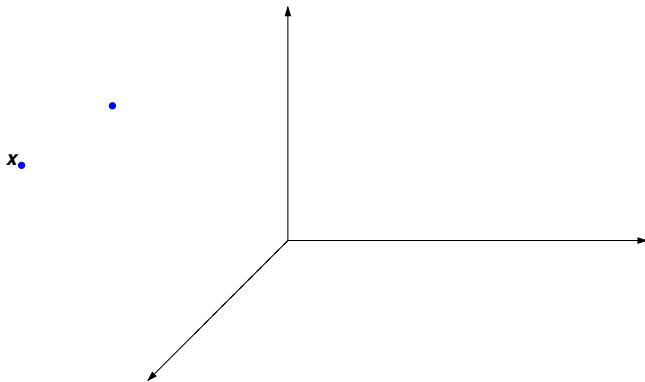
A 3D coordinate system is shown with three axes: a vertical axis pointing upwards, a horizontal axis pointing to the right, and a diagonal axis pointing towards the bottom-left. Each axis ends with an arrowhead. A point, labeled \mathbf{x}_0 , is marked with a small blue dot in the upper-left region of the coordinate system.

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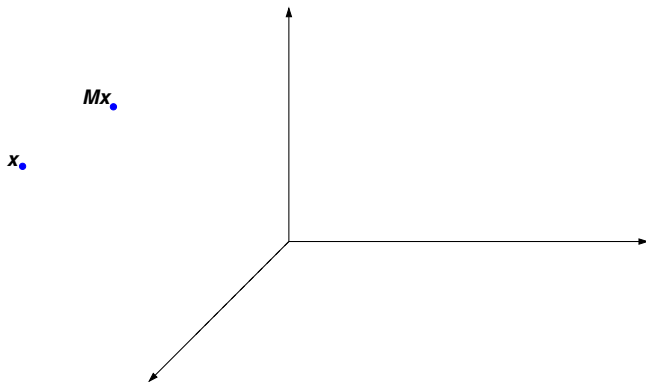


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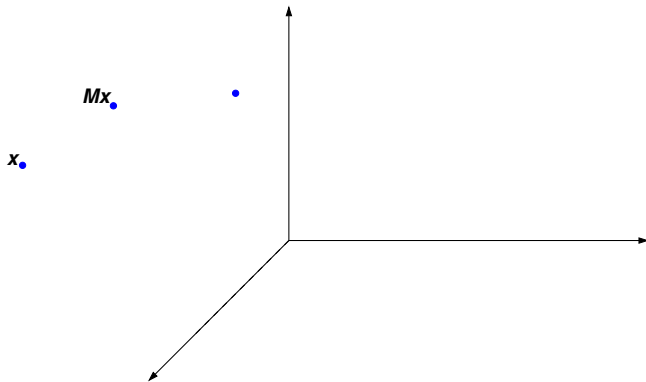


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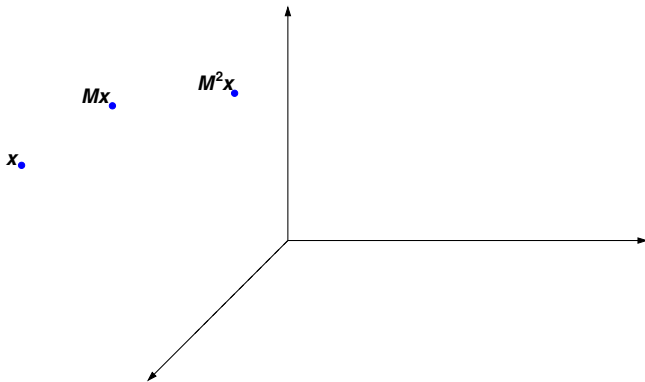


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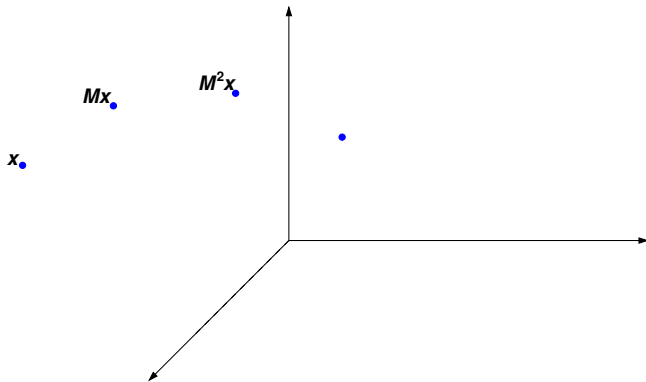


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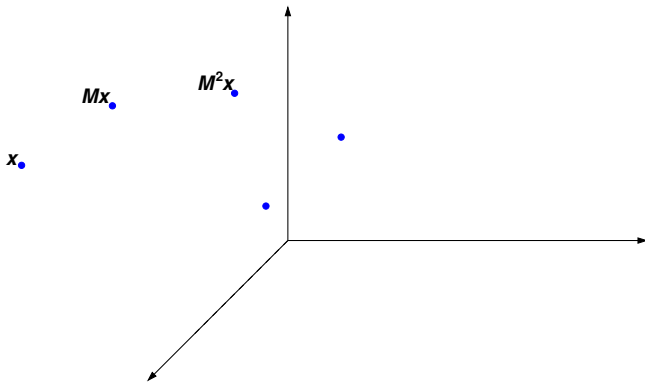


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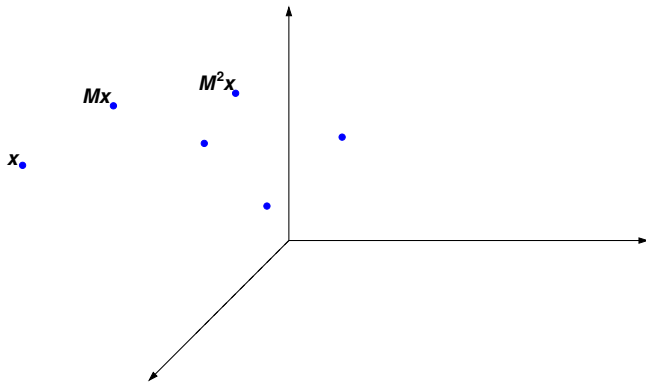


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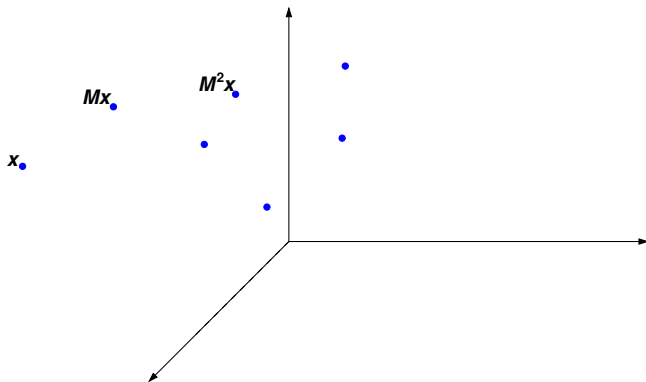


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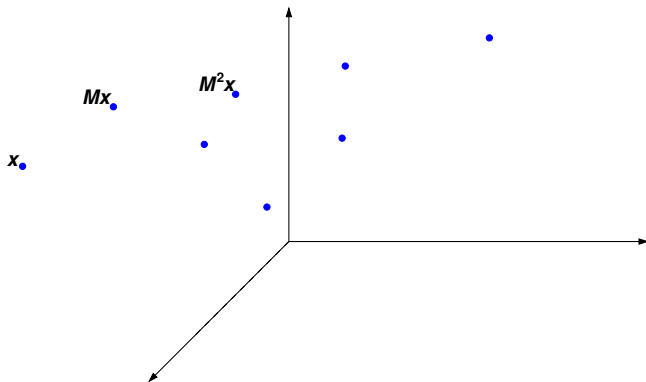


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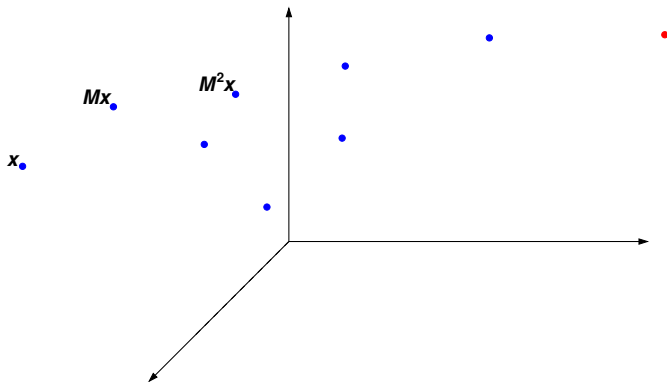


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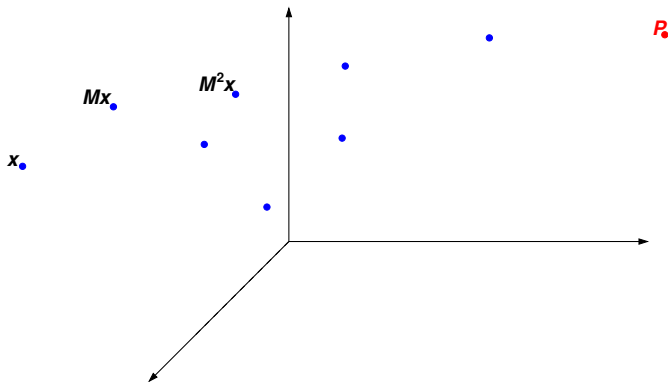


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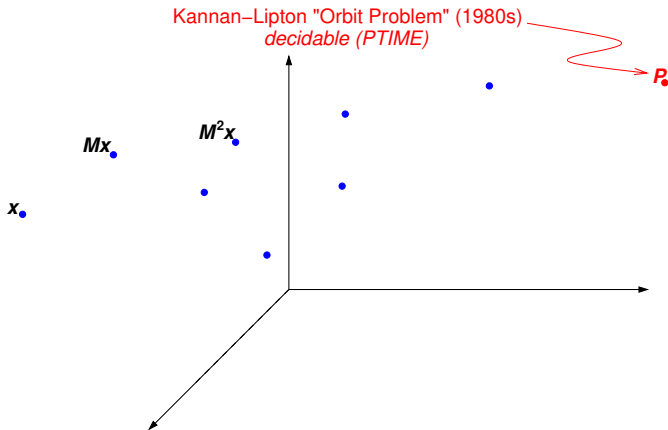


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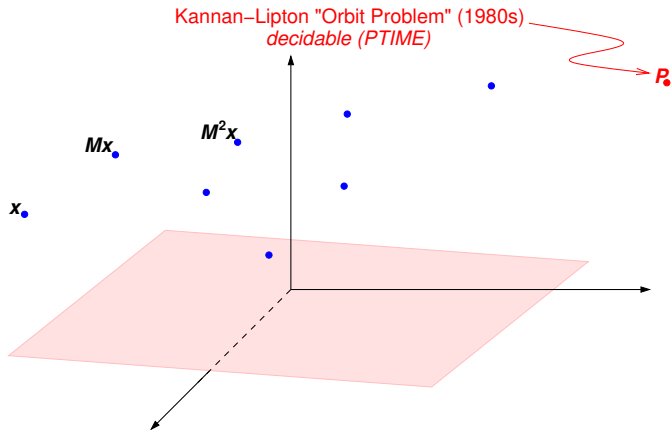


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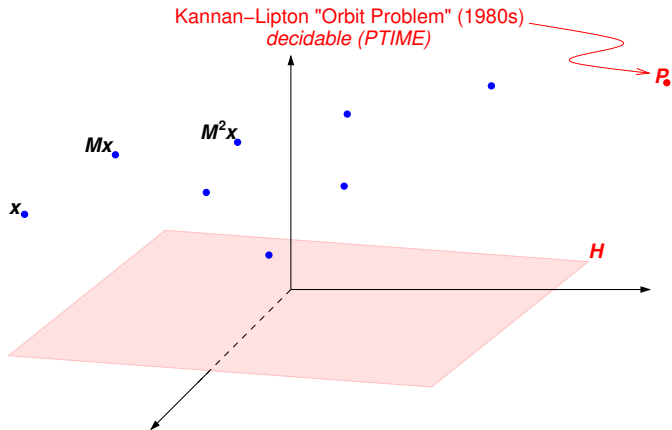


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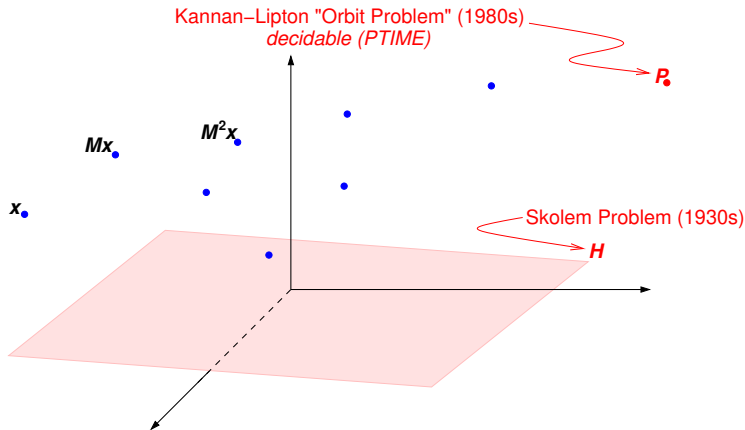


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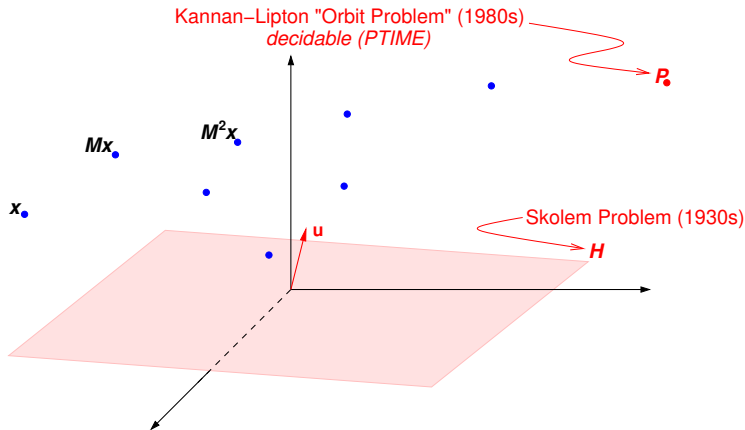


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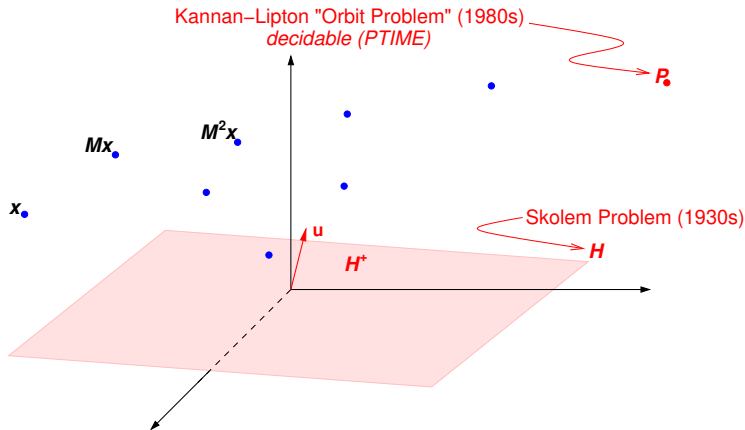


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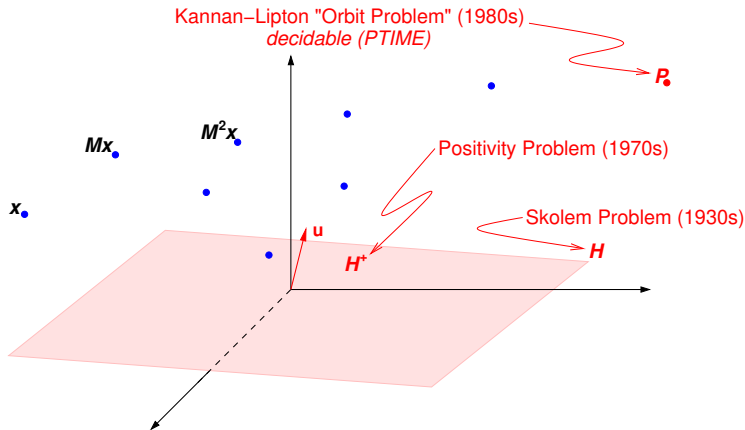


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Loop Termination

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x := 1;  
y := 0;  
z := 0;  
while x  $\neq$  0 do  
    x := 2x + y;  
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Some Other Application Areas

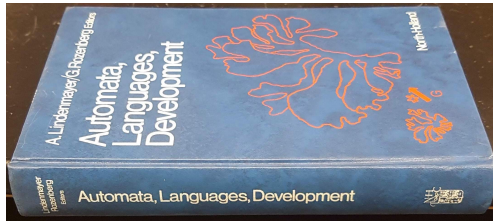
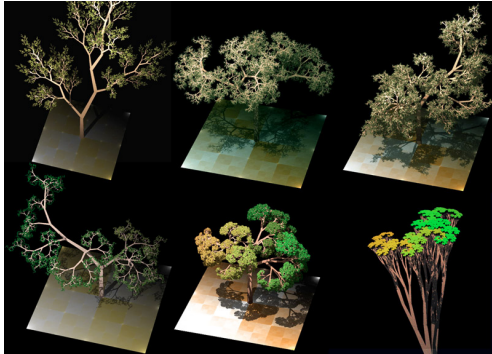
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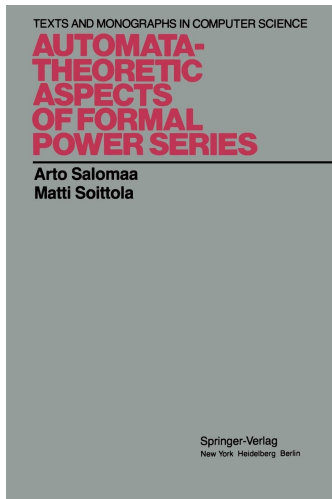
The Skolem and Positivity Problems arise in many other areas (often in hardness results), e.g.:

- theoretical biology (analysis of L-systems)
- software verification / program analysis
- differential privacy
- (weighted) automata and games
- analysis of stochastic systems
- control theory
- quantum computing
- statistical physics
- formal power series
- combinatorics
- ...

L-Systems (Aristid Lindenmayer, late 1960s)



Automata and Power Series (from 1960s, published 1978)



The Skolem and Positivity Problems: State of the Art

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Theorem (Mignotte, Shorey, Tijdeman 1984; Vereshchagin 1985)

In ambient dimension 4 or less, the Skolem Problem is decidable.

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Critical ingredient is Baker's theorem on linear forms in logarithms, which earned Baker the Fields Medal in 1970.



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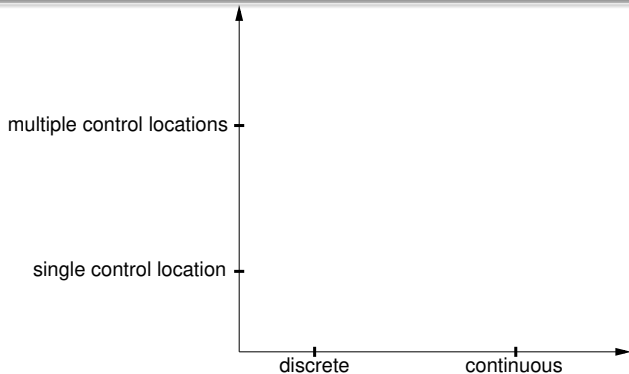
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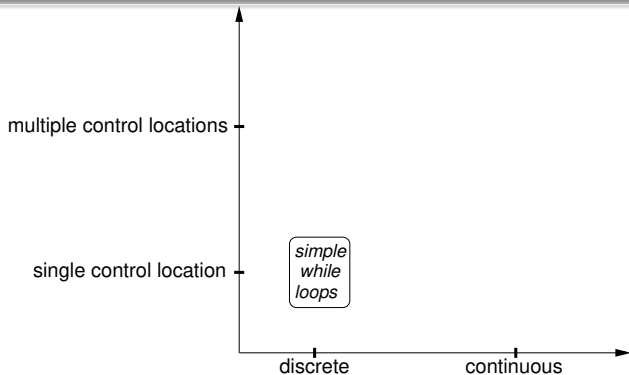
Theorem (O. & Worrell 2014)

- *In dimension ≤ 5 , Positivity is decidable.*
- *In dimension ≤ 9 , Positivity for diagonalisable matrices is decidable.*
- *In dimension ≥ 6 , Positivity is hard with respect to longstanding Diophantine-approximation problems.*

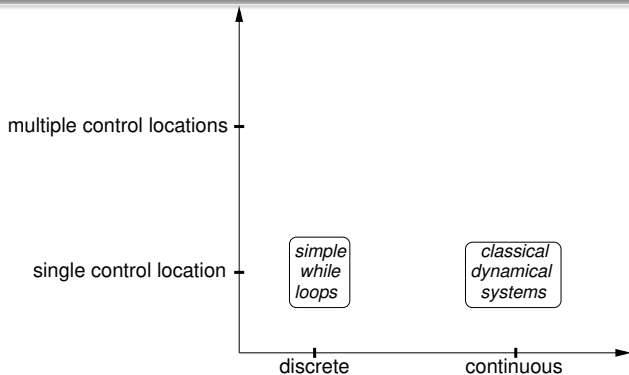
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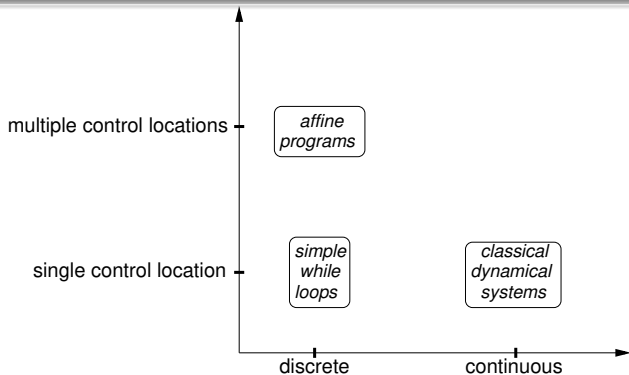
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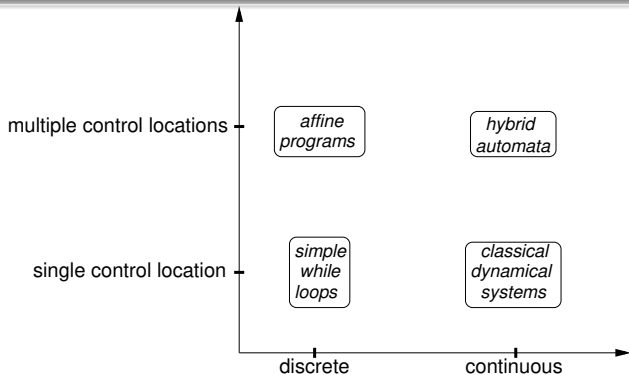
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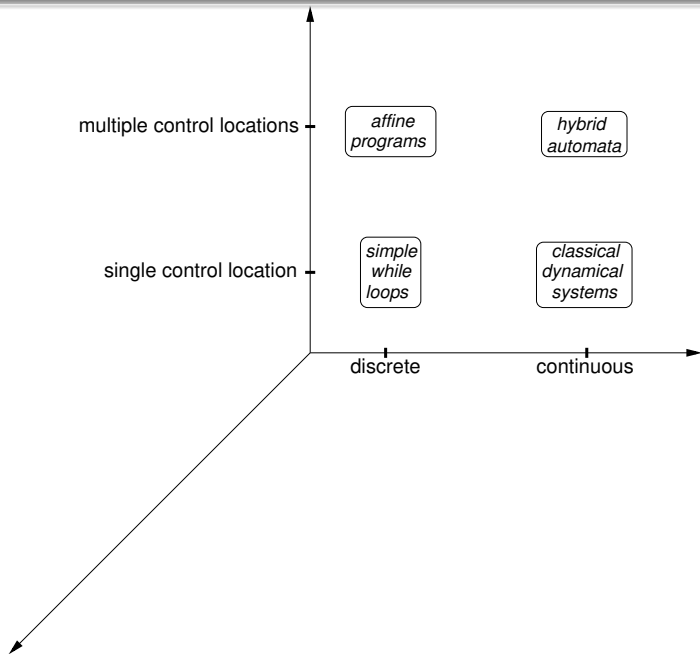
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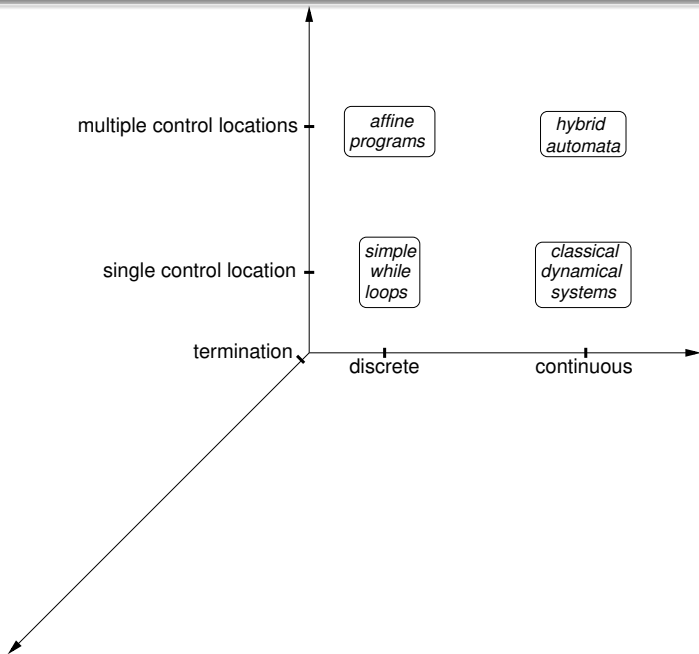
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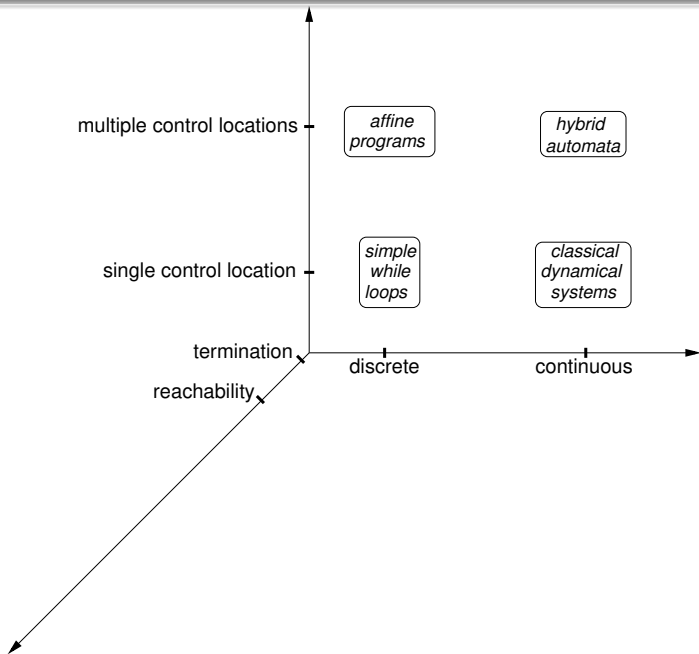
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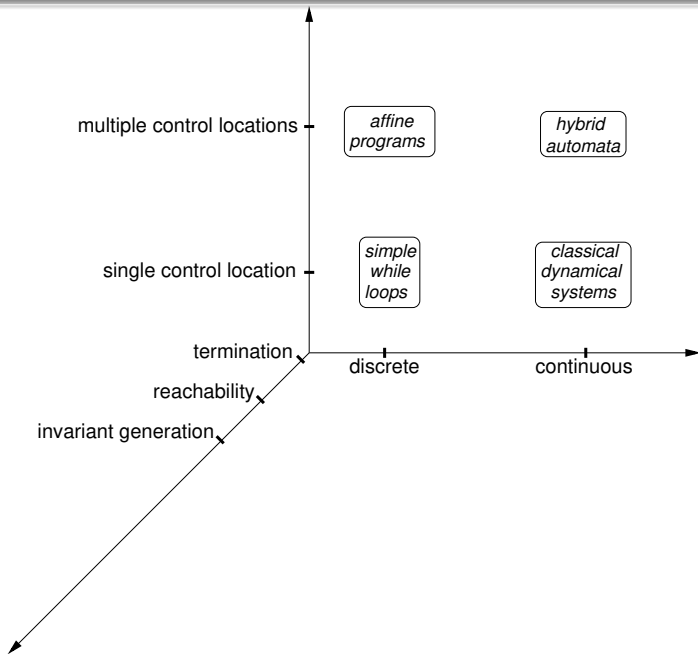
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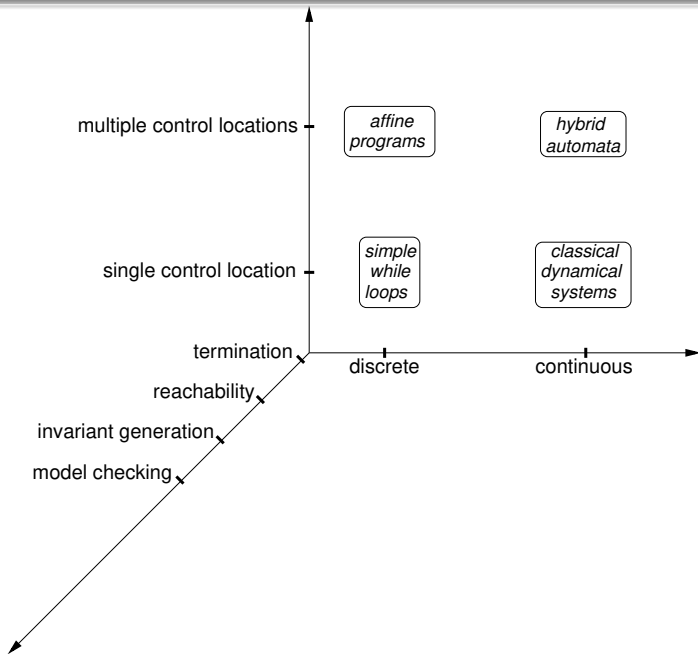
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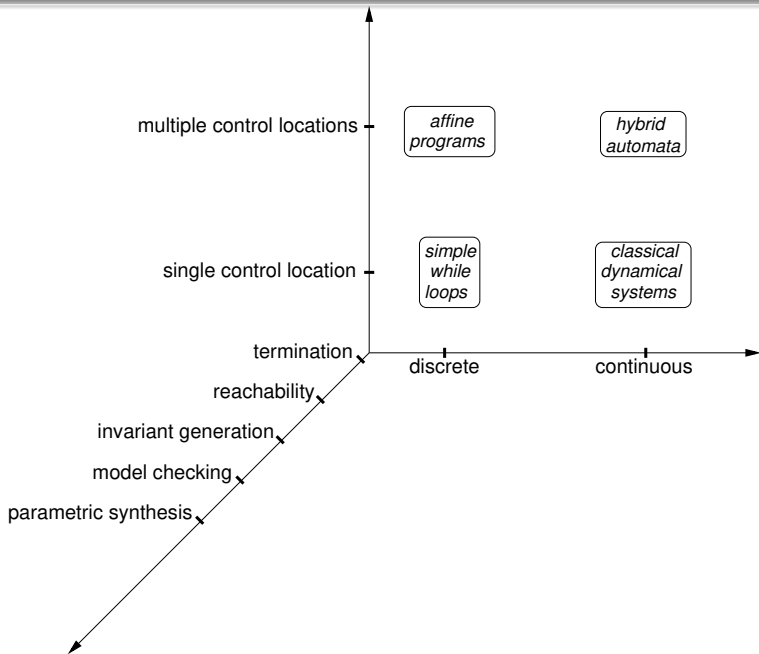
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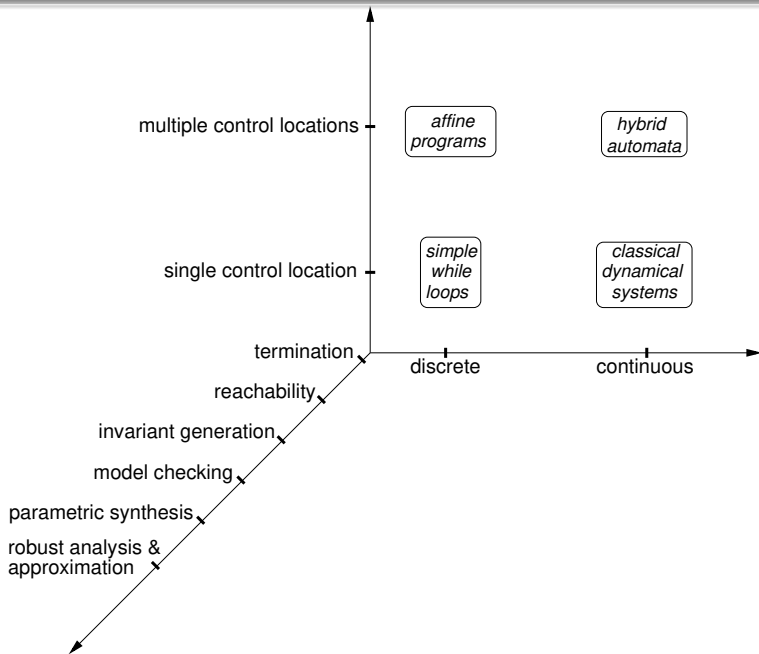
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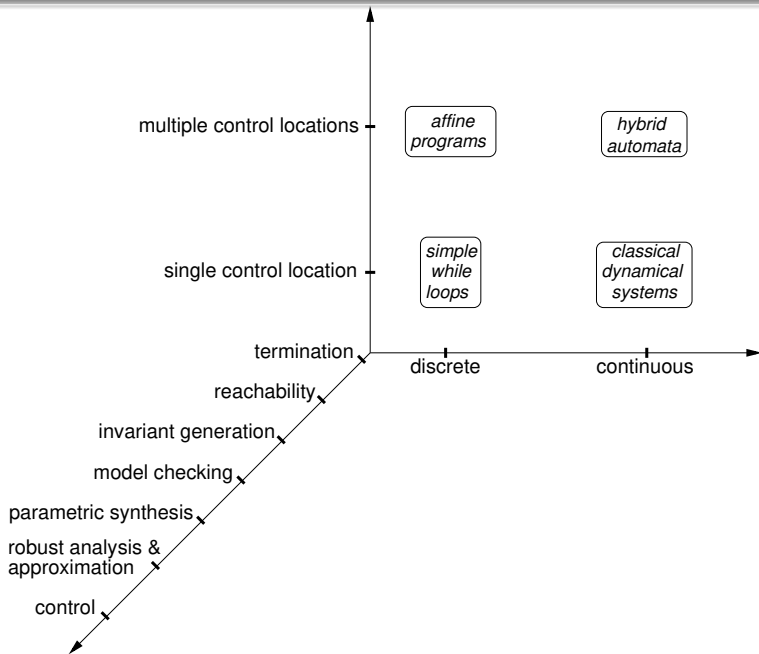
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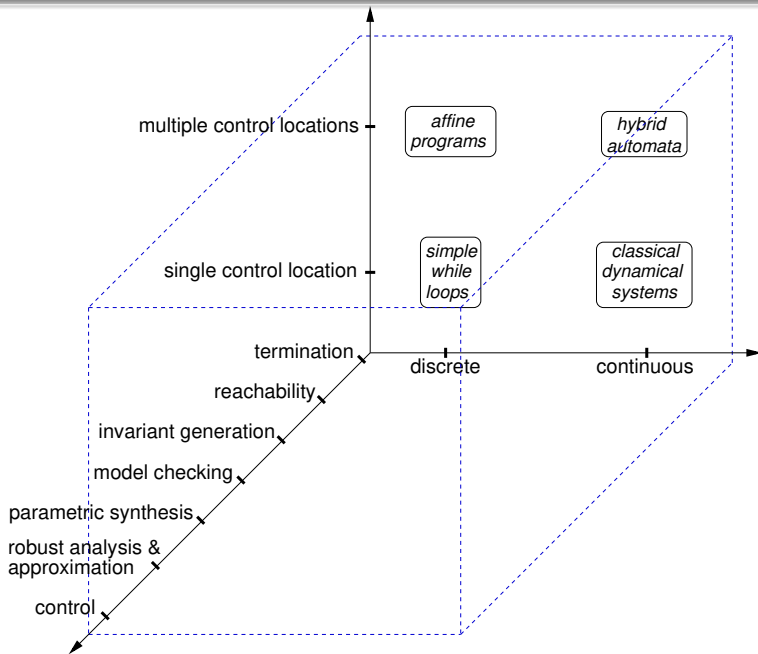
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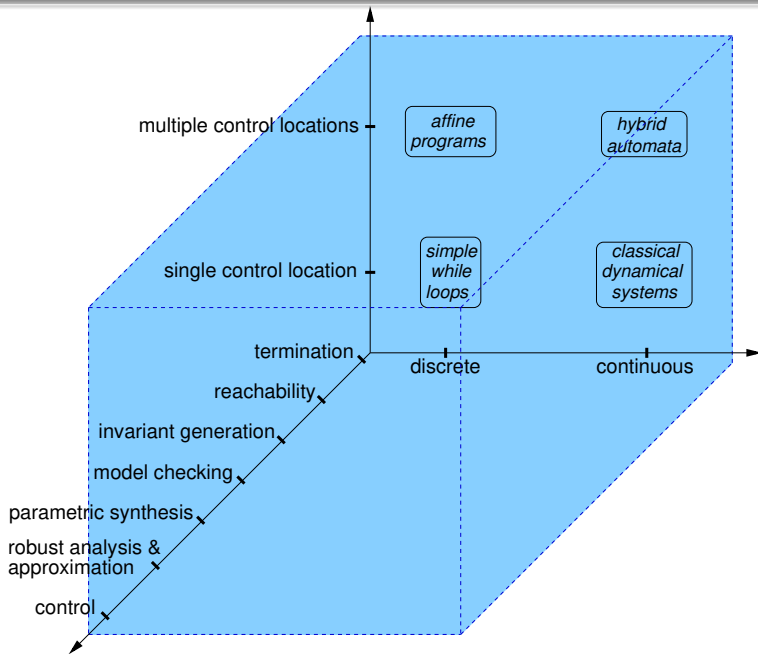
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