Exercise 2

1. We prove the result by induction on $u$:
   - if $u$ is equal to the empty transition $\varepsilon$ then the result is trivially obtained by taking $m_3 = m_2 = m_0$;
   - suppose now that $u = vt$ for some $v \in T^*$ and $t \in T$, we have $m_0 \xrightarrow{\omega} m \xrightarrow{t} m_2$ and by induction hypothesis we know that there exists $m'$ such that $m_0 \xrightarrow{v} m'$ and for all $p \notin \Omega(m')$ $m'(p) = m(p)$. $t$ is thus fireable from $m'$ and there exists a marking $m_3$ such that $m_0 \xrightarrow{v} m_3$ and for all $p \notin \Omega(m_3)$ we have $m_2(p) = m_3(p)$ (because $\Omega(m_3)$ contains $\Omega(m')$).

2. (a) We compute $\Theta(u)$ by induction: $\Theta(\varepsilon) = 0$ and $\Theta(tu) = \max(W(P, t), \Theta(u) + W(P, t) - W(t, P))$ where all operations are defined componentwise: indeed $\Theta(tu) \geq W(P, t)$ thus $t$ is enabled and leads to $\Theta(tu) = W(P, t) + W(t, P) \geq \Theta(u)$. By definition we must have $\Theta(u) + \Theta(v) \xrightarrow{u} m \geq \Theta(v)$ thus still allowing to fire $v$, hence $\Theta(uv) \leq \Theta(u) + \Theta(v)$.

   (b) Let $m''$ be the marking returned by $\text{AddOmeGas}(m, m', V)$. By definition of $\{v_1, \ldots, v_\ell\}$, the effect of each $v_i$ is:
   
   $\begin{align*}
   &= 0 \text{ for } p \in P \setminus \Omega(m''), \\
   &\geq 0 \text{ for } p \in \Omega(m'') \setminus \Omega(m) \text{ and } > 0 \text{ for at least one } p \in \Omega(m'') \setminus \Omega(m) \text{ unknown for } p \in \Omega(m).
   \end{align*}$

   The threshold required to fire $v_i$ on places $p$ in $P \setminus \Omega(m)$ is such that $\Theta(v_i)(p) \leq m''(p) \leq m'(p) = \nu_k(p)$ for the particular $m''_i$ of line 3 associated with $v_i$, and after firing $v_i$ we obtain a larger value in $p$, thus as far as these places are concerned $w^k$ can be fired.

   On the places in $\Omega(m)$, by question (a), the sequence $w^k$ can also be fired.

   Overall, we have

   $\nu'_k(p) \begin{cases} 
   = \nu_k(p) & \text{for } p \in P \setminus \Omega(m'') \\
   \geq k + \nu_k(p) & \text{for } p \in \Omega(m'') \setminus \Omega(m) \\
   \geq 0 & \text{for } p \in \Omega(m) .
   \end{cases}$

   (c) By induction on $u$ in $T^*$. For $u = \varepsilon$, $m_3 = m_0$, and $n = 0$ and $u_1 = \varepsilon$ fit: $m_0 \xrightarrow{\omega} m_0$.

   For the induction step, consider $m_0 \xrightarrow{u} m_3 \xrightarrow{t} m_3'$ and let $m_4 = \text{fire}(m_3, t)$, so that $m_3' = \text{AddOmeGas}(m_3, m_4, V)$.
If \( m'_3 = m_4 \) i.e. no \( \omega \) value was introduced, use the ind. hyp. on \( m_0 \xrightarrow{u} G m_3 \) with a partial marking \( \max(W(p, t), m'(p) + W(p, t) - W(t, p)) \) for all \( p \in \Omega(m_3) \). We obtain a run
\[
\begin{align*}
m_0 \xrightarrow{u_1w_1^{k_1}u_2w_2^{k_2}...u_nw_n^{k_n}u_{n+1}} \mathcal{N} m_2
\end{align*}
\]
with
\[
m_2(p) = \begin{cases} m_3(p) & \text{for all } p \in P \setminus \Omega(m_3) \\ \geq \max(W(p, t), m'(p) + W(p, t) - W(t, p)) & \text{for all } p \in \Omega(m_3). \end{cases}
\]

Thus \( t \) can be fired from \( m_2 \) and we have \( m_2 \xrightarrow{t} \mathcal{N} m'_2 \) with \( m'_2(p) = m'_3(p) \) for all \( p \in P \setminus \Omega(m_3) = P \setminus \Omega(m'_3) \) and
\[
m'_2(p) \geq \max(W(p, t), m'(p) + W(p, t) - W(t, p)) - W(p, t) + W(t, p) \geq m'(p)
\]
for all \( p \in \Omega(m_3) = \Omega(m'_3) \).

If \( m'_3 \neq m_4 \) i.e. \( \omega \) values were introduced on the places in \( \Omega(m'_3) \setminus \Omega(m_3) \).
Then, define
\[
k_{n+1} = \max_{p \in \Omega(m'_3) \setminus \Omega(m_3)} m'(p)
\]
\[
w_{n+1} = v_1 \cdots v_{\ell}
\]
the sequence associated with this particular invocation of \( \text{ADDOMEGAS}(m_3, m_4, V) \).
We use the ind. hyp. on \( m_0 \xrightarrow{u} G m_3 \) with a partial marking \( \nu_{k_{n+1}}(p) + W(p, t) + m'(p) \) for all \( p \in \Omega(m_3) \). We obtain a run of form \([1] \) with
\[
m_2(p) = \begin{cases} m_3(p) & \text{for all } p \in P \setminus \Omega(m_3) \\ \nu_{k_{n+1}}(p) + W(p, t) + m'(p) & \text{for all } p \in \Omega(m_3). \end{cases}
\]

Let us consider the transition sequence \( v = tw_{n+1}^{k_{n+1}} \): by questions (a) and (b), \( m_2 \) can fire \( v \) in \( \mathcal{N} \), leading to a marking \( m'_2(p) = m_2(p) - W(p, t) + W(t, p) = m_3(p) - W(p, t) + W(t, p) = m'_3(p) \) for all \( p \in P \setminus \Omega(m'_3) \), \( m'_2(p) \geq m'(p) \) for all \( p \in \Omega(m_3) \), and \( m'_2(p) \geq k_{n+1} \geq m'(p) \) for all \( p \in \Omega(m'_3) \setminus \Omega(m_3) \).