TD 10: Petri Nets
Solutions

Exercise 2

1. We prove the result by induction on u:

   - if u is equal to the empty transition ε then the result is trivially obtained by taking $m_3 = m_2 = m_0$;
   - suppose now that $u = vt$ for some $v \in T^*$ and $t \in T$, we have $m_0 \xrightarrow{v}_{\mathcal{N}} m \xrightarrow{t} m_2$ and by induction hypothesis we know that there exists m' such that $m_0 \xrightarrow{v}_G m'$ and for all $p \notin \Omega(m')$ $m'(p) = m(p)$. t is thus firable from m' and there exists a marking m_3 such that $m_0 \xrightarrow{u}_G m_3$ and for all $p \notin \Omega(m_3)$ we have $m_2(p) = m_3(p)$ (because $\Omega(m_3)$ contains $\Omega(m')$).

2. (a) We compute $\Theta(u)$ by induction: $\Theta(\varepsilon) = 0^P$ and $\Theta(tu) = \max(W(P,t), \Theta(u) + W(P,t) - W(t,P))$ where all operations are defined componentwise: indeed $\Theta(tu) \geq W(P,t)$ thus t is enabled and leads to $\Theta(tu) - W(P,t) + W(t,P) \geq \Theta(u)$. By definition we must have $\Theta(u)+\Theta(v) \xrightarrow{u} m \geq \Theta(v)$ thus still allowing to fire v, hence $\Theta(uv) \leq \Theta(u) + \Theta(v)$.

   (b) Let m'' be the marking returned by AddOmegas(m, m', V). By definition of $\{v_1, \ldots, v_\ell\}$, the effect of each v_i is

   $$= 0 \text{ for } p \in P \setminus \Omega(m''),$$
   $$\geq 0 \text{ for } p \in \Omega(m'') \setminus \Omega(m) \text{ and } > 0 \text{ for at least one } p \in \Omega(m'') \setminus \Omega(m)$$
   $$\text{unknown for } p \in \Omega(m)\ .$$

   The threshold required to fire v_i on places p in $P \setminus \Omega(m)$ is such that $\Theta(v_i)(p) \leq m''_i(p) \leq m'(p) = \nu_k(p)$ for the particular m''_i of line 3 associated with v_i, and after firing v_i we obtain a larger value in p, thus as far as these places are concerned w^k can be fired.

   On the places in $\Omega(m)$, by question (a), the sequence w^k can also be fired.

   Overall, we have

   $$\nu'_k(p) \begin{cases} = \nu_k(p) & \text{for } p \in P \setminus \Omega(m'') \\ \geq k + \nu_k(p) & \text{for } p \in \Omega(m'') \setminus \Omega(m) \\ \geq 0 & \text{for } p \in \Omega(m)\ . \end{cases}$$

   (c) By induction on u in T^*. For $u = \varepsilon$, $m_3 = m_0$, and $n = 0$ and $u_1 = \varepsilon$ fit: $m_0 \xrightarrow{\varepsilon}_{\mathcal{N}} m_0$.

   For the induction step, consider $m_0 \xrightarrow{u}_G m_3 \xrightarrow{t}_G m'_3$ and let $m_4 = \text{fire}(m_3, t)$, so that $m'_3 = $ AddOmegas(m_3, m_4, V).

If $m'_3 = m_4$ i.e. no ω value was introduced, use the ind. hyp. on $m_0 \xrightarrow{u}_G m_3$ with a partial marking $\max(W(p,t), m'(p) + W(p,t) - W(t,p))$ for all $p \in \Omega(m_3)$. We obtain a run

$$m_0 \xrightarrow{u_1 w_1^{k_1} u_2 \cdots u_n w_n^{k_n} u_{n+1}}_{\mathcal{N}} m_2 \tag{1}$$

with

$$m_2(p) \begin{cases} = m_3(p) \geq W(p,t) & \text{for all } p \in P \setminus \Omega(m_3) \\ \geq \max(W(p,t), m'(p) + W(p,t) - W(t,p)) & \text{for all } p \in \Omega(m_3). \end{cases}$$

Thus t can be fired from m_2 and we have $m_2 \xrightarrow{t}_{\mathcal{N}} m'_2$ with $m'_2(p) = m'_3(p)$ for all $p \in P \setminus \Omega(m_3) = P \setminus \Omega(m'_3)$ and

$$\begin{aligned} m'_2(p) &\geq \max(W(p,t), m'(p) + W(p,t) - W(t,p)) - W(p,t) + W(t,p) \\ &\geq m'(p) \end{aligned}$$

for all $p \in \Omega(m_3) = \Omega(m'_3)$.

If $m'_3 \neq m_4$ i.e. ω values were introduced on the places in $\Omega(m'_3) \setminus \Omega(m_3)$. Then, define

$$\begin{aligned} k_{n+1} &= \max_{p \in \Omega(m'_3) \setminus \Omega(m_3)} m'(p) \\ w_{n+1} &= v_1 \cdots v_\ell \end{aligned}$$

the sequence associated with this particular invocation of $\textsc{AddOmegas}(m_3, m_4, V)$. We use the ind. hyp. on $m_0 \xrightarrow{u}_G m_3$ with a partial marking $\nu_{k_{n+1}}(p) + W(p,t) + m'(p)$ for all $p \in \Omega(m_3)$. We obtain a run of form (1) with

$$m_2(p) \begin{cases} = m_3(p) \geq W(p,t) & \text{for all } p \in P \setminus \Omega(m_3) \\ \geq \nu_{k_{n+1}}(p) + W(p,t) + m'(p) & \text{for all } p \in \Omega(m_3). \end{cases}$$

Let us consider the transition sequence $v = t w_{n+1}^{k_{n+1}}$: by questions (a) and (b), m_2 can fire v in $\mathcal{N}$, leading to a marking m'_2 s.t. $m'_2(p) = m_2(p) - W(p,t) + W(t,p) = m_3(p) - W(p,t) + W(t,p) = m'_3(p)$ for all $p \in P \setminus \Omega(m'_3)$, $m'_2(p) \geq m'(p)$ for all p in $\Omega(m_3)$, and $m'_2(p) \geq k_{n+1} \geq m'(p)$ for all p in $\Omega(m'_3) \setminus \Omega(m_3)$.