TD 10: Petri Nets

Exercise 1 (Coverability Graph). The coverability problem for Petri nets is the following decision problem:

Instance: A Petri net $N = \langle P, T, F, W, m_0 \rangle$ and a marking $m_1$ in $\mathbb{N}^P$.

Question: Does there exist $m_2$ in $\text{reach}_N(m_0)$ such that $m_1 \leq m_2$?

For 1-safe Petri nets, coverability coincides with reachability, and is thus $\text{PSPACE}$-complete.

One way to decide the general coverability problem is to use Karp and Miller’s coverability graph (see the lecture notes). Indeed, we have the equivalence between the two statements:

i. there exists $m_2$ in $\text{reach}_N(m_0)$ such that $m_1 \leq m_2$, and

ii. there exists $m_3$ in $\text{CoverabilityGraph}_N(m_0)$ such that $m_1 \leq m_3$.

1. In order to prove that (i) implies (ii), we will prove a stronger statement: for a marking $m$ in $(\mathbb{N} \cup \{\omega\})^P$, write $\Omega(m) = \{p \in P \mid m(p) = \omega\}$ for the set of $\omega$-places of $m$.

Show that, if $m_0 \xrightarrow{u} N m_2$ in the Petri net $N$ for some $u$ in $T^*$, then there exists $m_3$ in $(\mathbb{N} \cup \{\omega\})^P$ such that $m_2(p) = m_3(p)$ for all $p$ in $P \setminus \Omega(m_3)$ and $m_0 \xrightarrow{u} G m_3$ in the coverability graph.

2. Let us prove that (ii) implies (i). The idea is that we can find reachable markings that agree with $m_3$ on its finite places, and that can be made arbitrarily high on its $\omega$-places. For this, we need to identify the graph nodes where new $\omega$ values were introduced, which we call $\omega$-nodes.

(a) The threshold $\Theta(u)$ of a transition sequence $u$ in $T^*$ is the minimal marking $m$ in $\mathbb{N}^P$ s.t. $u$ is enabled from $m$. Show how to compute $\Theta(u)$. Show that $\Theta(u \cdot v) \leq \Theta(u) + \Theta(v)$ for all $u, v$ in $T^*$.

(b) Recall that an $\omega$ value is introduced in the coverability graph thanks to Algorithm [1].

We consider a call to ADDOMegas$(m, m', V)$ on line 8 of the COVERABILITYGRAPH algorithm from the course notes, where $m \xrightarrow{t} N m'$ for $t$ the transition chosen at line 6 of the COVERABILITYGRAPH algorithm.

Let $\{v_1, \ldots, v_\ell\}$ be the set of “vt” sequences, where $v$ is found on line 3 of ADDOMegas$(m, m', V)$. These sequences vt resulted in adding at least one
repeat
  saved ← m′
  foreach m″ ∈ V s.t. ∃v ∈ T*, m″ ↛_G m do
    if m″ < m′ then
      m′ ← m′ + ((m′ − m″) · ω)
  end
until saved = m′;
return m′

Algorithm 1: ADDOMegas(m, m′, V)

\( \omega \) value to \( m′ \) on line 5. Let \( w = v_1 \cdots v_κ \). Show that, for any \( k \) in \( \mathbb{N} \), the marking \( ν_k \) defined by

\[
ν_k(p) = \begin{cases} 
  m′(p) & \text{if } p \in P \setminus \Omega(m) \\
  Θ(w_k)(p) & \text{if } p \in \Omega(m)
\end{cases}
\]

allows to fire \( w^k \). How does the marking \( ν'_k \) with \( ν_k \xrightarrow{w^k}_N ν'_k \) compare to \( ν_k \)?

(c) Prove that, if \( m_0 \xrightarrow{u}_G m_3 \) for some \( u \) in \( T^* \) in the coverability graph and \( m' \) in \( \mathbb{N}^{Ω(m_3)} \) is a partial marking on the places of \( Ω(m_3) \), then there are

- \( n \) in \( \mathbb{N} \),
- a decomposition \( u = u_1u_2 \cdots u_{n+1} \) with each \( u_i \) in \( T^* \) (where the markings \( µ_i \) reached by \( m_0 \xrightarrow{u_1 \cdots u_i}_G µ_i \) for \( i ≤ n \) have new \( \omega \) values),
- sequences \( w_1, \ldots, w_n \) in \( T^+ \),
- numbers \( k_1, \ldots, k_n \) in \( \mathbb{N} \),

such that \( m_0 \xrightarrow{u_1^{k_1}u_2^{k_2} \cdots u_n^{k_n}u_{n+1}}_N m_2 \) with \( m_2(p) = m_3(p) \) for all \( p \) in \( P \setminus Ω(m_3) \) and \( m_2(p) ≥ m′(p) \) for all \( p \) in \( Ω(m_3) \).

Exercise 2 (Decidability of Model-checking Action-based LTL).

1. Let \( N \) be Petri net, \( G \) its coverability graph, and \( m \) some marking in \( \mathbb{N}^P \). An infinite computation is a sequence \( m_0m_1 \cdots \in (\mathbb{N}^P)^\omega \) where for all \( i \in \mathbb{N} \), \( m_i \rightarrow_N m_{i+1} \) is a transition step. The effect \( Δ(u) \) of a transition sequence \( u \) in \( T^* \) is defined by \( Δ(ε) = 0^P \) and \( Δ(ut) = Δ(u) − W(P, t) + W(t, P) \).

Show that there exists an infinite computation s.t. \( m ≤ m_i \) for infinitely many indices \( i \) if and only if there exists an accessible loop \( m' \xrightarrow{v}_G m' \) in \( G \) s.t. \( m ≤ m' \) and \( Δ(v) ≥ 0^P \).

2. Show that action-based LTL model-checking is decidable for labeled Petri nets.
Exercise 3 (Rackoff’s Algorithm). A rather severe issue with the coverability graph construction is that it can generate a graph of Ackermannian size compared to that of the original Petri net. We show here a much more decent ExpSpace upper bound, which is matched by an ExpSpace hardness proof by Lipton.

Let us fix a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. We consider generalized markings in $\mathbb{Z}^P$. A generalized computation is a sequence $\mu_1 \cdots \mu_n$ in $(\mathbb{Z}^P)^*$ such that, for all $1 \leq i < n$, there is a transition $t$ in $T$ with $\mu_{i+1}(p) = \mu_i(p) - W(p, t) + W(t, p)$ for all $p \in P$ (i.e. we do not enforce enabling conditions). For a subset $I$ of $P$, a generalized sequence is $I$-admissible if furthermore $\mu_i(p) \geq W(p, t)$ for all $p$ in $I$ at each step $1 \leq i < n$. For a value $B$ in $\mathbb{N}$, it is $I$-B-bounded if furthermore $\mu_i(p) < B$ for all $p$ in $I$ at each step $1 \leq i \leq n$. A generalized sequence is an $I$-covering for $m_1$ if $\mu_1 = m_0$ and $\mu_n(p) \geq m_1(p)$ for all $p$ in $I$.

Thus a computation is a $P$-admissible generalized computation, and a $P$-admissible $P$-covering for $m_1$ answers the coverability problem.

For a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$ and a marking $m_1$ in $\mathbb{N}^P$, let $\ell(\mathcal{N}, m_1)$ be the length of the shortest $P$-admissible $P$-covering for $m_1$ in $\mathcal{N}$ if one exists, and otherwise $\ell(\mathcal{N}, m_1) = 0$. For $L, k$ in $\mathbb{N}$, define

$$M_L(k) = \sup\{\ell(\mathcal{N}, m_1) \mid |P| = k, \max_{p \in P, t \in T} W(p, t) + \max_{p \in P} m_1(p) \leq L\}$$

the maximal $\ell(\mathcal{N}, m_1)$ over all Petri nets $\mathcal{N}$ of dimension $k$ and all markings $m_1$ to cover, under some restrictions on incoming weights $W(p, t)$ in $\mathcal{N}$ and values in $m_1$.

1. Show that $M_L(0) \leq 1$.

2. We want to show that

$$M_L(k) \leq (L \cdot M_L(k-1))^k + M_L(k-1)$$

for all $k \geq 1$. To this end, we prove that, for every marking $m_1$ in $\mathbb{N}^P$ for a Petri net $\mathcal{N}$ with $|P| = k$,

$$\ell(\mathcal{N}, m_1) \leq (L \cdot M_L(k-1))^k + M_L(k-1). \quad (\ast)$$

Let

$$B = M_L(k-1) \cdot \max_{p \in P, t \in T} W(p, t) + \max_{p \in P} m_1(p).$$

and suppose that there exists a $P$-admissible $P$-covering $w = \mu_1 \cdots \mu_n$ for $m_1$ in $\mathcal{N}$.

(a) Show that, if $w$ is $P$-B-bounded, then $(\ast)$ holds.

(b) Assume the contrary: we can split $w$ as $w_1 w_2$ such that $w_1$ is $P$-B-bounded and $w_2$ starts with a marking $\mu_j$ with a place $p$ such that $\mu_j(p) \geq B$. Show that $(\ast)$ also holds.

3. Show that $M_L(|P|) \leq L^{(3|P|)!}$ for $L \geq 2$.\n
4. Given a Petri net $\mathcal{N} = (P, T, W, m_0)$ and a marking $m_1$, set $L = 2 + \max_{p \in P} m_1(p) + \max_{p \in P, t \in T} W(p, t)$. Assuming that the size $n$ of the instance $(\mathcal{N}, m_1)$ of the coverability problem is more than

$$\max(\log L, |P|, \max_{p \in P, t \in T} \log W(t, p)),$$

deduce that we can guess a $P$-admissible $P$-covering for $m_1$ of length at most $2^{c_n \log n}$ for some constant $c$. Conclude that the coverability problem can be solved in $\text{ExpSpace}$. 