TD 5: Emptiness Test for Büchi Automata, Partial-Order Reduction

Exercise 1

1. An inactive SCC in the explored graph contains only states such that you have a path from one to the other in $\mathcal{B}$, hence making it part of an SCC in $\mathcal{B}$. The question thus reduces to showing there is no other state in the corresponding SCC. Since the DFS search has ended for all states in the SCC, no other state can be reached from any of these, meaning it is indeed an SCC of $\mathcal{B}$.

2. By definition the roots of the SCCs in the active graph are visited before any other state in said SCC, hence if s belongs to the SCC rooted in r_i we have $r_i.num \leq s.num$. Similarly, if s belonged to some SCC rooted in r_j such that $j > i$ we would have $r_j.num \leq s.num$. This shows that if some state s is in the SCC rooted in r_i then we have $r_i.num \leq s.num$ and if $i < m$ then $s.num < r_{i+1}.num$. Since all states belong to one and only one SCC (possibly containing only that state), the mirror implication is also true.

3.

4. The algorithm returns *false* if and only if there exists some reachable state $u \in F$ such that there exists a non-trivial path from u to itself, which is equivalent to the non-emptiness of the language associated with the automaton.

5. Instead of keeping track only of the roots of the SCCs and said SCCs, we also keep track of the final sets that have a representant in the SCC (that is, we keep track of the (r, C, R) where $R \subseteq \{1, ...n\}$) is such that $i \in R$ if and only if there exists v in C such that $v \in F_i$). We then return *false* when we find an SCC containing a representant of each final set and a non-trivial path.

6. This algorithm considers every transition only once so it is faster than the nested DFS algorithm, that visits some transitions twice.

Exercise 2

1. Consider some $\mathrm{LTL}(\mathrm{AP}, \mathsf{U})$ formula φ. We will show by induction that for any stuttering equivalent words σ and ρ we have $\sigma \models \varphi \Leftrightarrow \rho \models \varphi$.

   - The cases $\varphi = \top, p \in \mathrm{AP}, \neg\psi, \psi_1 \vee \psi_2$ are trivial.
   - Consider then $\varphi = \psi_1 \mathsf{U} \psi_2$ and suppose that for any two stuttering equivalent words ζ and γ we have $\zeta \models \psi_1$ if and only if $\gamma \models \psi_1$ and $\zeta \models \psi_2$ if and only if $\gamma \models \psi_2$. Take two stuttering equivalent words σ and ρ. Consider then two sequences $0 = i_0 < i_1 < ...$ and $0 = j_0 < j_1 < ...$ such that for all l

we have $\sigma_{i_l} = ... = \sigma_{i_{l+1}-1} = \rho j_l = ... = \rho_{j_{l+1}-1}$. Now suppose there exists k such that $\sigma, k \models \psi_2$ and for all $k' < k$ we have $\sigma, k' \models \psi_1$. Consider l such that $i_l \leq k < i_{l+1}$. We can easily show that $\sigma_{\geq k}$ and ρ_{j_l} are stuttering equivalent, and hence $\rho, j_l \models \psi_2$. Similarly, for all n such that $n < j_l$ there exists $l' < l$ such that $\sigma_{\geq i_{l'}}$ and $\rho_{\geq n}$ are stuttering equivalent (l' is such that $j_{l'} \leq n < j_{l'+1}$ and $l' < l$), and since $\sigma, i_{l'} \models \psi_1$ we have $\rho, n \models \psi_1$. Hence we have shown $\sigma, 0 \models \varphi$ implies $\rho, 0 \models \varphi$ and since σ and ρ play a symmetrical role the mirror implication is also true.

2. Consider the sequence $i_0, i_1 ...$ defined as

   - $i_0 = 0$
   - for $k \geq 0$, i_{k+1} is the smallest number strictly greater than i_k such that $\sigma_{i_{k+1}} \neq \sigma_{i_k}$ if one such number exists, $i_k + 1$ else.

   Then σ' defined by $\sigma'_k = \sigma_{i_k}$ is the only stutter-free word that is stuttering equivalent to σ.

3. (a) If a stutter-free word σ is such that $\sigma, 0 \models a \wedge \mathsf{X}\, a$ then it is such that $\sigma_0 = a$ and $\sigma_1 = a$, hence by definition $\sigma = a^\omega$. Thus $\psi_{a,a} = \neg(\top \,\mathsf{U}\, \neg a) = \mathsf{G}\, a$ works.

   (b) The formula $\psi_{a,b} = a \wedge (a \,\mathsf{U}\, b)$ works.

4.
   - $\tau(\top) = \top$
   - $\tau(p) = p$ where $p \in \mathrm{AP}$
   - $\tau(\neg\psi) = \neg\tau(\psi)$
   - $\tau(\psi_1 \vee \psi_2) = \tau(\psi_1) \vee \tau(\psi_2)$
   - $\tau(\psi_1 \,\mathsf{U}\, \psi_2) = \tau(\psi_1) \,\mathsf{U}\, \tau(\psi_2)$
   - Notice that for stutter-free words, $a \wedge \mathsf{X}(b \wedge \psi)$ and $a \wedge (a \,\mathsf{U}\, (b \wedge \psi))$ are equivalent formula (see the previous question). Hence:

$$\tau(\mathsf{X}\,\psi) = \bigvee_{a \in \Sigma} \left((\mathsf{G}\, a \wedge \tau(\psi)) \vee \bigvee_{a \neq b} a \wedge (a \,\mathsf{U}\, (b \wedge \tau(\psi))) \right)$$

5. For any word σ, let $f(\sigma)$ be the only stutter-free word that is stuttering-equivalent to σ (as seen in question 3.). Then if $L(\varphi)$ is stutter-invariant we have $\sigma \models \varphi \Leftrightarrow f(\sigma) \models \varphi \Leftrightarrow f(\sigma) \models \tau(\varphi)$. However from question 1 we know that $L(\tau(\varphi))$ is stutter-invariant, and hence $f(\sigma) \models \tau(\varphi) \Leftrightarrow \sigma \models \tau(\varphi)$. This means $\sigma \models \varphi \Leftrightarrow \sigma \models \tau(\varphi)$ and thus $L(\varphi) = L(\tau(\varphi))$.