# TD 4
# Solutions

## Exercise 1

We construct the new automaton in the following way: we create $n+1$ "copies" of the automaton, having the same states and transitions for all non-accepting states. We number them from 0 to $n$. In copy number $k$, we call $s_k$ the state corresponding to $s \in Q$ in the original automaton. The set of initial states of the new automaton is the set $\{s_0 | s \in I\}$. Given a transition $s \to s'$ in the original automaton where $s \notin F_i$ for any $i$ we create transitions $s_k \to s'_k$ in all copies. Given a transition $s \to s'$ in the original automaton where $s \in F_i$ for some $i$, we create transitions $s_k \to s'_k$ for $k \neq i$ and a transition $s_i \to s'_{i+1}$ if $i \neq n$, $s_i \to s'_0$ else. Finally, the set of final states in the new automaton is the set $\{s_n | s \in F_n\}$.

The idea is that an execution in the new automaton goes from copy $i$ to copy $i+1$ when a state from $F_i$ is visited, hence it can visit the set of final states inifinitely many times if and only if the corresponding execution in the original automaton visits each $F_i$ infinitely often.

## Exercise 2

We first show that $\mathsf{Rec}(\Sigma^\omega) \subseteq \mathsf{Rat}(\Sigma^\omega)$. Let $\mathcal{A} = (Q, \Sigma, I, T, F)$ be a Büchi automaton recognizing some rational language $L$. For each $i \in I$, and $f \in F$, let $X_{i,f}$ be the language recognized by the finite-word automanton $(Q, \Sigma, \{i\}, T, \{f\})$ and $Y_f$ the language recognized by the finite-word automaton $(Q, \Sigma, \{f\}, T, \{f\})$. We then have $L = \cup_{(i,f) \in I \times F} X_{i,f} Y_f^\omega$ (the idea is that an accepting execution in $\mathcal{A}$ ends looping infinitely from some final state to itself).

To show that $\mathsf{Rat}(\Sigma^\omega) \subseteq \mathsf{Rec}(\Sigma^\omega)$, let us consider a rational language $L = \cup_{0 \leq i < n} X_i Y_i^\omega$. Consider finite-words automata $\mathcal{X}_i$ and $\mathcal{Y}_i$ that recognize $X_i$ and $Y_i$ respectevely. To construct a Büchi automaton recognizing $L$, we only need to connect final states in $\mathcal{X}_i$ to initial states in $\mathcal{Y}_i$, and final states in $\mathcal{Y}_i$ to initial states in $\mathcal{Y}_i$ for all $i$. The final states of our new automaton are final states in $\mathcal{Y}_i$ for some $i$.

## Exercise 3

1. The two automata are described in Figure 1 and Figure 2 respectively.

2. Suppose there exists some deterministic automaton $\mathcal{A}$ for $L$. Let $i$ be its initial state, $F$ its set of final states, and for any state $s$ of $\mathcal{A}$ and any word $w \in (a+b)^*$ let $\delta(s, w)$ be the state we end up in when reading $w$ from $s$ (notice this is always well-defined because with the help of a bin-state we can make the automaton complete). Since $a^\omega \in L$ there exists some $n_0 \in \mathbb{N}$ such that $\delta(i, a^{n_0}) \in F$. Similarly, since $a^{n_0} b a^\omega \in L$ there exists some $n_1$ such that $\delta(i, a^{n_0} b a^{n_1}) \in F$. By repeating this ad

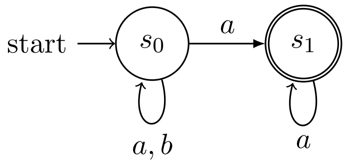


Figure 1: A nondeterministic Büchi automaton for $(a+b)^*a^\omega$.

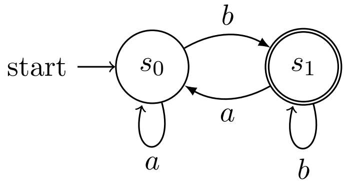


Figure 2: A deterministic Büchi automaton for $(a^*b)^\omega$.

infinitum, we construct a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that the word $a^{n_0}ba^{n_1}ba^{n_2}b...$, which does not belong to $L$, is accepted by $\mathcal{A}$.

3. We have $L' = \overrightarrow{L}$.

## Exercise 4

1. Each congruence class is associated with a unique function $Q^2 \to \{0,1,2\}$ mapping pairs of states $(q,q')$ to 2 if and only if all words $u$ in the congruence class satisfy $q \xrightarrow{u}_F q'$, 1 if and only if all words $u$ in the congruence class satisfy $q \xrightarrow{u} q'$ but not $q \xrightarrow{u}_F q'$ and 0 if no word $u$ in the class satisfies $q \xrightarrow{u} q'$. Two different congruence classes cannot be associated to the same function: this would mean the words in the two classes are equivalent to each other, and thus the two congruence classes are actually one and the same. $Q$ being finite, there is a finite number of functions $Q^2 \to \{0,1\}$ and hence $\mathcal{A}$ has a finite number of congruence classes.

2. For any pair of states $(q,q')$, we define by $\mathcal{L}(q,q')$ the language $\{u \in \Sigma^* | q \xrightarrow{u} q'\}$ and $\mathcal{L}_F(q,q')$ the language $\{u \in \Sigma^* | q \xrightarrow{u}_F q'\}$. Each of these languages is regular: $\mathcal{L}(q,q')$ is quite obviously recognized by the automaton $(Q,\Sigma,\{q\},T,\{q'\})$ and $\mathcal{L}(q,q')$ is recognized by a slightly more complicated automaton built from two copies of $\mathcal{A}$, with transitions from one copy to another from states in $F$, $q$ as the only initial state and the copy of $q'$ as the only accepting one. Each congruence class can be expressed as a finite intersection of languages of the form $\mathcal{L}_F(q,q')$, $\mathcal{L}(q,q')$ or the complement of one or the other. More precisely, for $u \in \Sigma^*$:

$$[u] = \bigcap_{u\in\mathcal{L}_F(q,q')} \mathcal{L}_F(q,q') \cap \bigcap_{\substack{u\in\mathcal{L}(q,q')\\ u\notin\mathcal{L}_F(q,q')}} \mathcal{L}(q,q') \cap \overline{\mathcal{L}_F(q,q')} \cap \bigcap_{u\notin\mathcal{L}(q,q')} \overline{\mathcal{L}(q,q')}$$

Hence, as a finite intersection of regular languages, each congruence class is itself a regular language.

3. As seen in exercise 2, since each congruence class is a regular language, each $[u][v]^\omega$ is a recongizable language. Since there are finitely many congruence classes, $K(L)$ is thus a finite union of recognizable languages, which means it is itself a recognizable language.

4. Consider some word $w \in K(L(\mathcal{A}))$. By design there exist words $u, v_0, v_1 \ldots \in \Sigma^*$ such that $w = uv_0v_1\ldots$ and there exists $v$ such that for all $i$ $v_i \in [v]$ and $[u][v]^\omega \cap L(\mathcal{A}) \neq \emptyset$. This means there exist words $u'$ and $v'_0, v'_1 \ldots$ such that $u \sim_\mathcal{A} u'$, for all $i$ $v_i \sim_\mathcal{A} v'_i$ and $u'v'_0v'_1 \ldots \in \mathcal{A}$. Consider an accepting execution $q_0q_1q_2...$ of $u'v'_0v'_1\ldots$ in $\mathcal{A}$. In particular there exist $k_0, k_1, ...$ such that $q_0 \xrightarrow{u'} q_{k_0}$, $q_{k_0} \xrightarrow{v'_0} q_{k_1}$ etc. Since this execution is accepting there exists $k_{i_0}, k_{i_1}...$ such that $q_{k_{i_0}} \xrightarrow{v'_{i_0-1}}_F q_{k_{i_0}+1}$, $q_{k_{i_1}} \xrightarrow{v'_{i_1-1}}_F q_{k_{i_1}+1}\ldots$ Since $u \sim_\mathcal{A} u'$ and for all $i$ $v_i \sim_\mathcal{A} v'_i$ we also have $q_0 \xrightarrow{u} q_{k_0}$, $q_{k_0} \xrightarrow{v_0} q_{k_1}$ etc., and for all $n$ $q_{k_{i_n}} \xrightarrow{v_{i_n-1}} q_{k_{i_n}+1}$, hence there exists an accepting execution of $uv_0v_1\ldots$ in $\mathcal{A}$, which means $w \in L(\mathcal{A})$.

5. For all $i, j$ with $i < j$, let $c(i,j)$ be the congruence class of $\sigma_i...\sigma_{j-1}$. As there is a finite number of congruence classes, we can apply Ramsey's theorem: there exists an inifinte set $A \subseteq \mathbb{N}$ and a congruence class $[v]$ such that for all $i, j \in A^2$ with $i < j$ we have $c(i,j) = [v]$. Let us order the elements in $A$: $A = \{i_n | n \in \mathbb{N}\}$ with for all $n$ $i_n < i_{n+1}$. We then have that for all $n$ $\sigma_{i_n}...\sigma_{i_{n+1}-1} \in [v]$. This means that $\sigma \in [\sigma_{<i_0}][v]^\omega$.

6. Using the previous question, we show that $\overline{L(\mathcal{A})} \subseteq K(\overline{L(\mathcal{A})})$, and thus $\overline{L(\mathcal{A})} = K(\overline{L(\mathcal{A})})$. Since $K(\overline{L(\mathcal{A})})$ is recognizable, this means $\overline{L(\mathcal{A})}$ is recognizable.