Exercise 1

We construct the new automaton in the following way: we create \( n+1 \) “copies” of the automaton, having the same states and transitions for all non-accepting states. We number them from 0 to \( n \). In copy number \( k \), we call \( s_k \) the state corresponding to \( s \in Q \) in the original automaton. The set of initial states of the new automaton is the set \( \{ s_0 | s \in I \} \). Given a transition \( s \rightarrow s' \) in the original automaton where \( s \notin F_i \) for any \( i \) we create transitions \( s_k \rightarrow s'_k \) in all copies. Given a transition \( s \rightarrow s' \) in the original automaton where \( s \in F_i \) for some \( i \), we create transitions \( s_k \rightarrow s'_k \) for \( k \neq i \) and a transition \( s_i \rightarrow s'_{i+1} \) if \( i \neq n \), \( s_i \rightarrow s'_{0} \) else. Finally, the set of final states in the new automaton is the set \( \{ s_n | s \in F_n \} \).

The idea is that an execution in the new automaton goes from copy \( i \) to copy \( i+1 \) when a state from \( F_i \) is visited, hence it can visit the set of final states infinitely many times if and only if the corresponding execution in the original automaton visits each \( F_i \) infinitely often.

Exercise 2

We first show that \( \text{Rec}(\Sigma^\omega) \subseteq \text{Rat}(\Sigma^\omega) \). Let \( A = (Q, \Sigma, I, T, F) \) be a Büchi automaton recognizing some rational language \( L \). For each \( i \in I \), and \( f \in F \), let \( X_{i,f} \) be the language recognized by the finite-word automaton \( (Q, \Sigma, \{i\}, T, \{f\}) \) and \( Y_f \) the language recognized by the finite-word automaton \( (Q, \Sigma, \{f\}, T, \{f\}) \). We then have \( L = \bigcup_{(i,f) \in I \times F} X_{i,f} Y_f^\omega \) (the idea is that an accepting execution in \( A \) ends looping infinitely from some final state to itself).

To show that \( \text{Rat}(\Sigma^\omega) \subseteq \text{Rec}(\Sigma^\omega) \), let us consider a rational language \( L = \bigcup_{0 \leq i < n} X_i Y_i^\omega \). Consider finite-words automata \( X_i \) and \( Y_i \) that recognize \( X_i \) and \( Y_i \) respectively. To construct a Büchi automaton recognizing \( L \), we only need to connect final states in \( X_i \) to initial states in \( Y_i \), and final states in \( Y_i \) to initial states in \( Y_i \) for all \( i \). The final states of our new automaton are final states in \( Y_i \) for some \( i \).

Exercise 3

1. The two automata are described in Figure 1 and Figure 2 respectively.

2. Suppose there exists some deterministic automaton \( A \) for \( L \). Let \( i \) be its initial state, \( F \) its set of final states, and for any state \( s \) of \( A \) and any word \( w \in (a+b)^* \) let \( \delta(s, w) \) be the state we end up in when reading \( w \) from \( s \) (notice this is always well-defined because with the help of a bin-state we can make the automaton complete).

Since \( a^\omega \in L \) there exists some \( n_0 \in \mathbb{N} \) such that \( \delta(i, a^{n_0}) \in F \). Similarly, since \( a^{n_0}ba^\omega \in L \) there exists some \( n_1 \) such that \( \delta(i, a^{n_0}ba^{n_1}) \in F \). By repeating this ad
Figure 1: A nondeterministic Büchi automaton for \((a + b)^*a^\omega\).

![Figure 1: A nondeterministic Büchi automaton for \((a + b)^*a^\omega\).]

Figure 2: A deterministic Büchi automaton for \((a^*b)^\omega\).

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infinitum, we construct a sequence of natural numbers \((n_k)_{k \in \mathbb{N}}\) such that the word \(a^{n_0}ba^{n_1}ba^{n_2}b\ldots\), which does not belong to \(L\), is accepted by \(\mathcal{A}\).

3. We have \(L' = \overline{L}\.\)

Exercise 4

1. Each congruence class is associated with a unique function \(Q^2 \to \{0, 1, 2\}\) mapping pairs of states \((q, q')\) to 2 if and only if all words \(u\) in the congruence class satisfy \(q \xrightarrow{u} F q'\), 1 if and only if all words \(u\) in the congruence class satisfy \(q \xrightarrow{u} q'\) but not \(q \xrightarrow{u} F q'\) and 0 if no word \(u\) in the class satisfies \(q \xrightarrow{u} q'\). Two different congruence classes cannot be associated to the same function: this would mean the words in the two classes are equivalent to each other, and thus the two congruence classes are actually one and the same. Since \(Q\) being finite, there is a finite number of functions \(Q^2 \to \{0, 1\}\) and hence \(\mathcal{A}\) has a finite number of congruence classes.

2. For any pair of states \((q, q')\), we define by \(L(q, q')\) the language \(\{u \in \Sigma^* | q \xrightarrow{u} q'\}\) and \(L_F(q, q')\) the language \(\{u \in \Sigma^* | q \xrightarrow{u} F q'\}\). Each of these languages is regular: \(L(q, q')\) is quite obviously recognized by the automaton \((Q, \Sigma, \{q\}, T, \{q'\})\) and \(L_F(q, q')\) is recognized by a slightly more complicated automaton built from two copies of \(\mathcal{A}\), with transitions from one copy to another from states in \(F\), \(q\) as the only initial state and the copy of \(q'\) as the only accepting one. Each congruence class can be expressed as a finite intersection of languages of the form \(L_F(q, q')\), \(L(q, q')\) or the complement of one or the other. More precisely, for \(u \in \Sigma^*:\)

\[
[u] = \bigcap_{u \in L_F(q, q')} L_F(q, q') \cap \bigcap_{u \notin L_F(q, q')} \overline{L_F(q, q')} \cap \bigcap_{u \notin L(q, q')} \overline{L(q, q')}
\]
Hence, as a finite intersection of regular languages, each congruence class is itself a regular language.

3. As seen in exercise 2, since each congruence class is a regular language, each $[u][v]^\omega$ is a recognizable language. Since there are finitely many congruence classes, $K(L)$ is thus a finite union of recognizable languages, which means it is itself a recognizable language.

4. Consider some word $w \in K(L(A))$. By design there exist words $u, v_0, v_1 \ldots \in \Sigma^*$ such that $w = uv_0v_1\ldots$ and there exists $v$ such that for all $i$ $v_i \in [v]$ and $[u][v]^\omega \cap L(A) \neq \emptyset$. This means there exist words $u'$ and $v'_0, v'_1 \ldots$ such that $u \sim_A u'$ for all $i$ $v_i \sim_A v'_i$ and $u'v'_0v'_1\ldots \in A$. Consider an accepting execution $q_0q_1q_2\ldots$ of $u'v'_0v'_1\ldots$ in $A$. In particular there exist $k_0, k_1, \ldots$ such that $q_0 \xrightarrow{u'} q_{k_0}$, $q_{k_0} \xrightarrow{v'_0} q_{k_1}$ etc. Since this execution is accepting there exists $k_{i_0}, k_{i_1}, \ldots$ such that $q_{k_{i_0}} \xrightarrow{v'_{i_0-1}} q_{k_{i_0}+1}$, $q_{k_{i_1}} \xrightarrow{v'_{i_1-1}} q_{k_{i_1}+1}$\ldots Since $u \sim_A u'$ and for all $i$ $v_i \sim_A v'_i$ we also have $q_0 \xrightarrow{u} q_{k_0}$, $q_{k_0} \xrightarrow{v_0} q_{k_1}$ etc., and for all $n$ $q_{k_{i_n}} \xrightarrow{v_{i_n-1}} q_{k_{i_n}+1}$, hence there exists an accepting execution of $uv_0v_1\ldots$ in $A$, which means $w \in L(A)$.

5. For all $i, j$ with $i < j$, let $c(i, j)$ be the congruence class of $\sigma_i \ldots \sigma_{j-1}$. As there is a finite number of congruence classes, we can apply Ramsey’s theorem: there exists an infinite set $A \subseteq \mathbb{N}$ and a congruence class $[v]$ such that for all $i, j \in A^2$ with $i < j$ we have $c(i, j) = [v]$. Let us order the elements in $A$: $A = \{i_n | n \in \mathbb{N}\}$ with for all $n$ $i_n < i_{n+1}$. We then have that for all $n$ $\sigma_{i_n} \ldots \sigma_{i_{n+1}-1} \in [v]$. This means that $\sigma \in \sigma_{< i_0}[v]^\omega$.

6. Using the previous question, we show that $L(A) \subseteq K(L(A))$, and thus $L(A) = K(L(A))$. Since $K(L(A))$ is recognizable, this means $L(A)$ is recognizable.