TD 3: CTL, CTL* 

Exercise 1 (Equivalences). Are the following formulæ equivalent?
1. $A X A G \varphi$ and $A X G \varphi$
2. $E X E G \varphi$ and $E X G \varphi$
3. $A (\varphi \land \psi)$ and $A \varphi \land A \psi$
4. $E (\varphi \land \psi)$ and $E \varphi \land E \psi$
5. $\neg A (\varphi \Rightarrow \psi)$ and $E (\varphi \land \neg \psi)$

Exercise 2 (Semantics of CTL*). Compute $\llbracket \varphi \rrbracket$, where:

\[ M \begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \rightarrow 4 \rightarrow 5
\end{array}
\]

$\varphi = A[(Xq) \lor FA((EFGp)U(AGq))]$

Exercise 3 (CTL Model-Checking). Let $M = (S, T, I, AP, \ell)$ be a finite Kripke structure, and $\varphi$ a CTL formula.
1. Let $M_\varphi$ be the restriction of $M$ to states satisfying $\varphi$: $M_\varphi = (\llbracket \varphi \rrbracket, T \cap \llbracket \varphi \rrbracket^2, I \cap \llbracket \varphi \rrbracket, AP, \ell|_{\llbracket \varphi \rrbracket})$.
   Show that $s \in \llbracket EG \varphi \rrbracket$ iff there exists a non-trivial strongly connected component $C$ of $M_\varphi$ and $t \in C$ such that $s \rightarrow^* t$ in $M_\varphi$.
2. Deduce an algorithm to compute $\llbracket EG \varphi \rrbracket$ from $M$ and $\llbracket \varphi \rrbracket$. What is the complexity of your procedure?

Exercise 4 (CTL+). CTL+ extends CTL by allowing boolean connectives on path formulæ, according to the following abstract syntax:

\[
\begin{align*}
f & ::= \top \mid a \mid f \land g \mid \neg f \mid E \varphi \mid A \varphi \quad \text{(state formulæ $f, g$)} \\
\varphi & ::= \varphi \land \psi \mid \neg \varphi \mid X f \mid f U g \quad \text{(path formulæ $\varphi, \psi$)}
\end{align*}
\]

where $a$ is an atomic proposition. The associated semantics is that of CTL*.

We want to prove that, for any CTL+ formula, there exists an equivalent CTL formula.
1. Give an equivalent CTL formula for
\[ E((a_1 \cup b_1) \land (a_2 \cup b_2)). \]

2. Generalize your translation for any formula of form
\[ E \left( \bigwedge_{i=1,\ldots,n} (\psi_i \cup \psi'_i) \land G \varphi \right). \] (1)

What is the complexity of your translation?

3. Give an equivalent CTL formula for the following CTL\(^+\) formula:
\[ E(Xa \land (b \cup c)). \]

4. Using subformulæ of form (1) and \( E \) modalities, give an equivalent CTL formula to
\[ E(X\varphi \land \bigwedge_{i=1,\ldots,n} (\psi_i \cup \psi'_i) \land G \varphi'). \] (2)

What is the complexity of your translation?

5. We only have to transform any CTL\(^+\) formula into (nested) disjuncts of form (2).
Detail this translation for the following formula:
\[ A((F a \lor X a \lor X \neg b \lor F \neg d) \land (d \cup \neg c)). \]