TD 2: Temporal Logics
Solutions

Exercice 1

First, we define the predicate $t' = t + 1$ for two variables t, t': $t' = t + 1 := (t < t') \wedge \forall t''(t < t'' \rightarrow (t' = t'' \vee t' < t''))$. We then define the predicate $t' = t + 2$: $t' = t + 2 := \exists t''(t'' = t + 1 \wedge t' = t'' + 1)$.

1. (a) $\mathsf{G}((y = 1) \rightarrow \neg\,\mathsf{X}(y = 1))$
   (b) $\forall t((y = 1)(t) \rightarrow \neg(\exists t', (y = 1)(t') \wedge t' = t + 1))$
2. (a) $\mathsf{G}((x = 1) \rightarrow \mathsf{X}(y = 1) \vee \mathsf{X}\,\mathsf{X}(y = 1))$
   (b) $\forall t((x = 1)(t) \rightarrow (\exists t'(y = 1)(t') \wedge (t' = t + 1 \vee t' = t + 2)))$
3. (a) $\mathsf{G}(x = 1 \rightarrow ((r = 0 \rightarrow \mathsf{X}(r = 0)) \wedge (r = 1 \rightarrow \mathsf{X}(r = 1))))$
   (b) $\forall t((x = 1)(t) \rightarrow \forall t'(t' = t + 1 \rightarrow (((r = 0)(t) \rightarrow (r = 0)(t')) \wedge ((r = 1)(t) \rightarrow (r = 1)(t')))))$
4. (a) $\mathsf{G}\,\mathsf{F}\,r = 1$
   (b) $\forall t \exists t'(t < t' \wedge (r = 1)(t'))$

Exercice 2

1. We have $\neg q \,\mathsf{R}\, q \Leftrightarrow \mathsf{G}\, q$, hence φ_2 implies φ_1, but not the other way around.
2. The formula is equivalent to $\top$: by unfolding we obtain $\mathsf{SF}(p) \vee \mathsf{SF}(\neg p) \vee \ldots$. One can also see that if the right part is not satisfied then $\mathsf{SF}(p)$ is satisfied, which satisfies the left part.
3. There are three possible cases for this formula to be satisfied:
   - either r holds right away
   - or just before r holds q holds
   - or just before r holds p holds and q does not, but q will hold later (and until then p will hold in an uninterrupted fashion).

   This means this formula is equivalent to $r \vee (p \vee q)\mathsf{U}(q \wedge (q \mathsf{U} r)) \vee (p \vee q)\mathsf{U}(r \wedge (p \mathsf{U} q))$.

Exercice 3

1. (a) $p \mathsf{U} \mathsf{G} \neg p$

   (b) There are at least two meaningful answers:
   - Basic, from scratch: $p \wedge \mathsf{X} p \wedge \mathsf{X}\mathsf{X} p ... \wedge \mathsf{X}^{n-1} p \wedge \mathsf{X}^n \mathsf{G} \neg p$
   - Reusing the previous question: $(p \mathsf{U} \mathsf{G} \neg p) \wedge X^{n-1} p \wedge X^n G \neg p$

2. This language is defined by the formula $p \wedge \mathsf{G}(p \to \mathsf{X} \neg p) \wedge \mathsf{G}(\neg p \to \mathsf{X} p)$.

3. We want to prove by induction the property $\mathcal{P}(\varphi)$: given n the number of X modalities in φ, for all $i, i' > n$, $\sigma_i \models \varphi$ iff $\sigma_{i'} \models \varphi$. The only tricky case is when $\varphi = \varphi_1 \mathsf{U} \varphi_2$, which we detail here. We suppose $\mathcal{P}(\varphi_1)$ and $\mathcal{P}i(\varphi_2$. Given n the number of X modalities in φ, we want to show that for all $i > n$, $\sigma_i \models \varphi$ iff $\sigma_{n+1} \models \varphi$. Let $i > n$ and suppose $\sigma_{n+1} \models \varphi$. Then

   (1) $\sigma_{n+1} \models \varphi$ or

   (2) there exists $j < n+1$ such that $\sigma_j \models \varphi_2$ and for all k such that $j < k \leq n+1$ we have $\sigma_k \models \varphi_1$.

   If (1) holds then $\mathcal{P}(\varphi_2)$ immediately yields $\sigma_i \models \varphi_2$ and hence $\sigma_i \models \varphi$ (because there are at most as many X modalities in φ_2 as in φ). If (2) holds then by $\mathcal{P}(\varphi_1)$ we have that for all $k \geq n+1$ $\sigma_k \models \varphi_1$. Thus for all k such that $j < k \leq i$ we have $\sigma_i \models \varphi_1$, and $\sigma_j \models \varphi_2$, which means that $\sigma_i \models \varphi$. The case $\sigma_i \models \varphi \Rightarrow \sigma_{n+1} \models \varphi$ is similar.

4. If this set were expressible there would exist a formula to express it. This formula would have a fixed number of X modalities, which by the previous question would mean it would not be satisfied by all the words in the set or would be satisfied by some words that are not in the set.

Exercice 4

1. We define $\tilde{\varphi}$ by induction:
   - $\tilde{\top} := \top$
   - $\tilde{q} := q$
   - $\widetilde{\neg\varphi_1} := \neg\tilde{\varphi}_1$
   - $\widetilde{\varphi_1 \vee \varphi_2} := \tilde{\varphi}_1 \vee \tilde{\varphi}_2$
   - $\widetilde{\varphi_1 \mathsf{U} \varphi_2} := \tilde{\varphi}_1 \mathsf{U} \tilde{\varphi}_2$
   - $\widetilde{\varphi_1 \mathsf{SS} \varphi_2} := \tilde{\varphi}_1 \mathsf{SS} (\tilde{\varphi}_2 \wedge \mathsf{SP} p)$

2. We let $\bar{\varphi} = \neg p \mathsf{U} p \wedge \tilde{\varphi}$.