Exercise 1  Total population
Let $\mathcal{Y}$ be the total population of the branching process whose probability of extinction is 1, given by $\mathcal{Y} = \sum_{i,n} Z_i^{(n)}$. Let $g_\mathcal{Y}$ be its generating function.
Show that $g_\mathcal{Y}(s) = s g_Z(g_\mathcal{Y}(s))$.

Exercise 2  Extinction rate and concentration of the total population
Assume that $\mathbb{E}[Z] > 1$. We denote $T$ the extinction time of the Galton-Watson process with offspring law $Z$. $T$ can then be infinite and we denote $p_{\text{ext}}$ the extinction probability of the process.

1. Show that $p_{\text{ext}} - \mathbb{P}(T \leq n) \leq g_Z'(p_{\text{ext}})^n$.
2. Give the expression in the particular case of $Z \sim \text{Poi}(\lambda)$, with $\lambda > 1$.

We now assume that $\mathbb{E}[Z] < 1$, and are interested in the total population (which corresponds to the extinction time if at each step one node generates its children). We note $\mathcal{Y}$ the total population.

3. Bound $\mathbb{P}(\mathcal{Y} > n)$ with a Chernoff bound.
4. Give an explicit bound in the case where $Z \sim \text{Poi}(\lambda)$, with $\lambda < 1$.

Exercise 3  Branching process conditioned on extinction
The history of a process is given by the sequence $H = \{Z_1, Z_1, \ldots, Z_T\}$ of the number of children in a one-by-one exploration: $Y_t > 0$ for all $t < T$, and $Y_T = 0$.

1. Consider $x_1, \ldots, x_T$ a finite history. Express $\mathbb{P}(H = (x_1, \ldots, x_k))$ in function of the distribution of $Z$.

In the remaining, $Z$ has a Poisson distribution with parameter $\lambda > 1$. As a consequence, its extinction probability is $p_{\text{ext}} < 1$. Let $\phi(s) = e^{s-1}$.

Define $\mu = \lambda p_{\text{ext}}$.

2. Show that $\mu$ is the only solution of $\frac{\phi(\lambda)}{\lambda} = \frac{\phi(s)}{s}$ and $s < 1$.
3. Show that conditioned on extinction, the distribution of the histories coincides with the distribution of the histories under a Poisson offspring distribution with parameter $\mu$.

Exercise 4  Branching processes in continuous time
Consider the following process:
— at time 0, $Z_0 = 1$ (the root of the process). By convention, this node is born at time 0.
— when a node $i$ is born, its lifetime has an exponentially distribution with parameter $\mu$: if it is born at time $t$, it dies at time $t + U_i$, with $U_i$ exponentially distributed with parameter $\mu$.
— a live node $i$ can give birth to children. Children are generated according to an exponentially distribution with parameter $\lambda$: if a node is born at time $t$, its first child (if it is not dead before) is generated at time $t + V_i^{(1)}$, the second child at time $t + V_i^{(1)} + V_i^{(2)}$, and so on where $V_i^{(j)}$ is exponentially distributed with parameter $\lambda$. 
— all the lifetimes \((U_i)\) and birth interval \((V_{i}^{(j)})\) for a mutually independent family of
random variables.

We recall that for \(X\) an exponentially distributed random variable with parameter \(\lambda\) satisfies:
\[
\forall t \geq 0, \quad P(X \geq t) \leq e^{-\lambda t}.
\]

1. Show that the exponential distribution is memory-less: if \(X\) is exponentially distributed
with parameter \(\lambda\), \(\forall t, u \geq 0\),
\[
P(X \geq t + u \mid X \geq t) = P(X \geq u).
\]

Let \(X_1\) and \(X_2\) be two independent exponentially distributed random variables with respec-
tive parameters \(\lambda\) and \(\mu\).

2. Show that \(\min(X_1, X_2)\) is also exponentially distributed. What is its parameter?

3. What is the probability that \(\min(X_1, X_2) = X_1\)?

We are back to the branching process.

4. What is the law of the number of children for each node?

5. What is the probability of extinction of this process?