The probabilistic method

Exercice 1 **Independence and intersection**

Consider two disjoint events with non-null probability. Can they be independent ?

Exercice 2 **Independence, mutual independence**

Give an example of a probability space with the events A_1, A_2 and A_3 that are two-by-two independent, but not mutually independent.

Exercice 3 **Fix points of a permutation**

What are the expectation and variance of the number of fix points of a uniformly distributed permutation on $\{1, \dots, n\}$?

Exercice 4 **Size of an independent set of a graph**

Let $G = (V, E)$ be a finite non directed graph. An independent set I is a subset of vertices no two elements of I are adjacent :

$$I \text{ independent set } \Rightarrow \forall u, v \in I,\ (u, v) \notin E.$$

Set $\alpha(G)$ the maximal size of an independent set of G, and $d(v)$ the degree of vertex v.

We build a graph G at random the following way : vertices are added one by one in a uniformly distributed order.

1. What is the probability that a vertex v is added before all its neighbors ?

2. Deduce that $\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}$.

Exercice 5 **Dominating set of a graph**

Let is first recall a useful formula : $(1 - p) \leq e^{-p}$ for all $p \in [0, 1]$.

A dominating set D is a set of vertives such that every other vertex is adjacent to a vertex of D :

$$D \text{ dominating set } \Rightarrow \forall v \notin D,\ \exists u \in D \text{ such that } (u, v) \in E.$$

1. Show that if G is k-regular (all its vertices have degree exactly k), then for all dominating set, D, $|D| \geq \frac{|V|}{k+1}$.

We now assume that all the vertices of G have degree at least k, snd we build a dominating set the following way :

1. Select each vertex of V in a set S with probability $p = \frac{\ln(k+1)}{k+1}$;
2. T is the subset of the vertices in $V \setminus S$ that have no neighbors in S ;
3. $D = S \cup T$.

2. Show that D is a dominating set.

3. What is the probability that a vertex v belongs to T ?

4. Deduce that there exists a dominating set of size at most $|V| \frac{1+\ln(k+1)}{k+1}$.

`Exercice 6` `Distinct sums`

A set of integers A is call a *distinct-sum* set if the sum $\sum_{a\in S} a$ is different fo each subset of S of A. For all non-negative integer n, we define $f(n)$ as the maximal size of a distinct-sum subset of $\{1, 2, \dots, n\}$.

1. Show that $f(n) \geq 1 + \lfloor \log_2 n \rfloor$ (give an example of distinct-sum subset).

2. Using a counting argument, show that $f(n) \leq \log_2 n + \log_2 \log_2 n + O(1)$.

The goal of the rest of the exercise is to improve the coefficient of the term $\log_2 \log_2 n$ by showing that

$$f(n) \leq \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1) \ .$$

Let $A \subseteq \{1, 2, \dots, n\}$ be a distinct-sum set. Set $k = |A|$ and X a subset of A chosen uniformly at random, and define S_X as the sum of the elements of X.

3. Let $\lambda > 1$. Show that

$$\mathbf{P}\big(\left|S_X - \mathbf{E}[S_X]\right| \geq \lambda n\sqrt{k}/2\big) \leq 1/\lambda^2.$$

4. Deduce that

$$n \geq \frac{2^k(1 - 1/\lambda^2) - 1}{\sqrt{k}\lambda} \ .$$

5. Conclude.