Probabilistic Aspects of Computer Science: TD3
Random walks and applications

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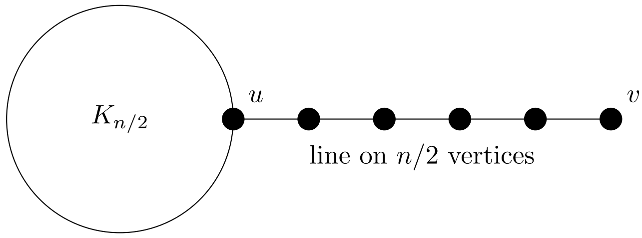
Exercise 1 (Cover time)**.** Let $G = (V, E)$ be a finite, undirected, connected graph without self-loops and with at least two vertices. A random walk on G is a Markov chain M defined by the sequence of moves of a particle between vertices of G. In this process, the place of the particle at a given time step is the state of the system. If the particle is at vertex i and if i has $d(i)$ outgoing edges, the probability that the particle follows the edge $\{i, j\}$ and moves to a neighbor j is $1/d(i)$.

1. Show that a random walk on an undirected graph G is aperiodic if and only if G is not bipartite.
2. In the rest of the exercise, we assume that G is not bipartite. Show that a random walk on G converges to a steady-state distribution π, where $\pi_v = \frac{d(v)}{2|E|}$.
3. We consider a new Markov chain M' defined on the edges of G. The current state in M' is defined to be the pair composed of the edge most recently traversed in the random walk in M, together with the direction of this traversal: the state space is hence the set of directed edges. How many states does M' have? Describe its transition matrix $\mathbf{Q}$.
4. Let $\{u, v\} \in E$. What is (in M') the mean return time $\mu_{(u,v),(u,v)}$ to the directed edge (u, v)?
5. Let $\{u, v\} \in E$. Starting at u, what is the expected waiting time $\mu_{u,(v,u)}$ before taking the directed edge (v, u)?
6. We denote $\mu_{v,u}$ the expected number of steps to reach u from v. Show that if $\{u, v\} \in E$, then $\mu_{u,v} + \mu_{v,u} \leq 2|E|$.
7. Starting from an arbitrary vertex in an arbitrary connected, undirected $G = (V, E)$, what is the minimal number of steps sufficient to visit all the vertices and come back to the original vertex? (Give a formula depending on $|V|$.)
8. The *cover time* of G, which we denote C_G, is defined as the maximum over all vertices $v \in V$ of the expected time to visit all of the nodes in the graph by a random walk starting from v. Show that the cover time of G is bounded above by $2|E|(|V| - 1)$.
9. As an application, suppose we are given an undirected graph $G = (V, E)$ and two vertices s and t in G, and we want to determine whether there is a path connecting s and t. For simplicity, assume that the graph G has no bipartite connected components. By standard deterministic search algorithms, we can easily solve the problem in linear time, using $\Omega(n)$ space. Show that the following algorithm returns the correct answer with probability $1/2$, and it only errs by returning that there is no path from s to t when there is such a path. What is the time and space complexities of this algorithm? (*Hint: you may use the Markov's inequality, which says for a random variable X and $a > 0$ that* $\mathbf{Pr}(|X| \geq a) \leq \frac{\mathbf{E}(|X|)}{a}$.)

> ***s*-*t* Connectivity algorithm**
> 1. Start a random walk from s.
> 2. If the walk reached t within $2|V|^3$ steps, return that there is a path. Otherwise, return that there is no path.

Exercise 2 (Computation of cover times)**.**

1. What is the cover time of a line when we start on an end node of the line?
2. Consider a complete graph (without loops) with n vertices named $1, \dots, n$. Let $k < n$. Starting from vertex n, what is the expected waiting time before visiting some vertex at most k?
3. What is the cover time of a complete graph without loops?
4. The lollipop graph on n vertices is a clique on $n/2$ vertices connected with a line on $n/2$ vertices as shown below:

   The node u is a part of both the clique and the line. Let v denote the other end of the line.

   Show that the cover time of a random walk starting at v is $\Theta(n^2)$.
5. Show that the cover time of a random walk starting at u is $\Theta(n^3)$.

Exercise 3 (Cat and mouse)**.** A cat and a mouse each independently take a random walk on a connected, undirected, non-bipartite graph G, with n vertices and m edges. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Show an upper bound of $O((nm)^2)$ on the expected time before the cat eats the mouse. What is a good strategy for the cat to eat quickly the mouse?