Probabilistic Aspects of Computer Science: TD2
Markov chains in the long run

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Exercise 1 (Move-to-front heuristic)**.** Consider $n \geq 1$ records $R_1, R_2, \ldots, R_n$ that can be ordered in any order. The cost of accessing the jth record in the order is j. For example, with four records ordered as R_2, R_4, R_3, R_1, the cost of accessing R_4 is 2 and the cost of accessing R_1 is 4.

Assume that at every discrete time the record R_j is accessed with fixed positive probability p_j.

1. If we know the values of the p_j, how should we order the records to minimize the expected cost? What would be this cost?

In the remainder we assume that we do not know the p_j. To keep the cost low we use a *move-to-front* heuristic: at each step, put the record that was accessed at the front of the list (to the left). We assume that moving the record can be done at no cost and that the other records remain in the same order. For example, if the order is R_2, R_4, R_3, R_1 before R_3 is accessed, it becomes R_3, R_2, R_4, R_1 afterwards.

2. Describe this problem by a Markov chain $(X_k)_k$ whose state space is the set of words where the name or index of each register occurs exactly once. Describe $P = M(p_1, \ldots, p_n)$ its transition probability matrix. Show that it admits a unique steady-state distribution.

3. The purpose of this question is to determine the steady-state distribution π of $(X_k)_k$. For a word w and a letter a, let $w \setminus a$ be the result of removing all a's from w.

   (a) Assume that $n \geq 2$. Let $\sigma = \sigma_1 \ldots \sigma_n$ be an ordering of the registers. Express π_σ using other orderings.

   (b) Assume that $n \geq 2$. What is $(X_k \setminus \sigma_1)_k$? Describe it qualitatively and quantitatively by a matrix Q.

   (c) Find a relation between π_σ and $\pi^Q_{\sigma\setminus\sigma_1}$, where π^Q is the steady-state distribution of Q

   (d) Assume that $n \geq 2$. Find a relation between Q and $P' := M(\frac{p_{\sigma_2}}{1-p_{\sigma_1}} \ldots \frac{p_{\sigma_n}}{1-p_{\sigma_1}})$, and between their eigenvectors.

   (e) Let
   $$\varphi_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{x_i}{\sum_{j=i}^{n} x_j}$$
   Show that for all σ we have $\pi_\sigma = \varphi_n(p_{\sigma_1}, \ldots, p_{\sigma_n})$ (the steady-state distribution of the chain).

4. The purpose of this question is to determine the expected cost of the move-to-front method. Let C_k be the cost for accessing the kth requested record, $V_{k,i}$ be the event that the k-th query refers to record R_i, and $W_{k,i}$ the Boolean random variable that equals 1 iff $V_{k,i}$ holds. Using the law of total expectation we obtain the following equality.
   $$\mathbf{E}(C_k) = \sum_{i=1}^{n} \mathbf{E}(C_k \mid V_{k,i}) \mathbf{Pr}(V_{k,i})$$

   (a) For all k and $i \neq j$ let $L_{k,j,i} := 1$ if the register R_j is ordered before R_i just before the kth request, and $L_{k,j,i} := 0$ otherwise. Under some assumption, find a relation between C_k and the $L_{k,j,i}$.

(b) Are $V_{k,i}$ and $L_{k,j,i}$ dependent or independent?

(c) Find a relation between $\mathbf{E}(C_k \mid V_{k,i})$ and the $L_{k,j,i}$.

(d) Describe the behavior of $\mathbf{E}(L_{k,j,i})$ when k approaches $+\infty$.

(e) Describe the behavior of $\mathbf{E}(C_k)$ when k approaches $+\infty$, and compare with the first question.

Exercise 2. We revisit an exercise from last week using a new and more direct tool.

1. Let X_n be the number of heads obtained after n independent tosses of a (possibly unfair) coin. Show that, for any $k \geq 2$,
$$\lim_{n\to\infty} \mathbf{Pr}(X_n \text{ is divisible by } k) = \frac{1}{k}$$

2. Solve the problem when X_n represents the sum of n independent rolls of a dice.

Exercise 3. Exhibit a Markov chain which has null recurrent states (different from the one studied during the lecture).

Exercise 4. Show that if a Markov chain has two steady-state distributions, then it has an infinite number of steady-state distributions.