Probabilistic Aspects of Computer Science: TD1
Discrete-time Renewal Processes and First Markov Chains

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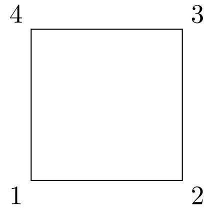
Exercise 1. Consider a person walking on the following square.



The person starts at intersection 1. Then, at every intersection, the person tosses a fair coin and if the coin turns up heads then the person moves anti-clockwise, otherwise the person moves clockwise.

1. Describe the situation with a finite discrete-time Markov chain, with transition matrix $\mathbf{P}$. Give the transition graph of the Markov chain. What is the initial distribution π_0?

   The states are $S = \{1, 2, 3, 4\}$ and the transition matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

   We let $\pi_0 = (1, 0, 0, 0)$.

2. Compute the distribution $\pi_n = \pi_0 \, \mathbf{P}^n$ for all natural numbers n. Does this Markov chain admit a steady-state distribution?

   We have $\pi_1 = (0, \frac{1}{2}, 0, \frac{1}{2})$ and $\pi_2 = (\frac{1}{2}, 0, \frac{1}{2}, 0) = \pi_0$, hence $\pi_n = \pi_0$ if n is even and π_1 if n is odd. The limit $\lim_{n\to\infty} \pi_n$ does not exist, which means that there is no steady-state distribution.

3. Now, consider the same example with different rules. Instead of tossing one faire coin, a person tosses two fair coins. If the first coin turns up heads then the person decides to stay in the position it was before; otherwise the person tosses another coin. If the second coin turns up heads, the person moves anti-clockwise, otherwise the person moves clockwise. Give the new transition matrix $\mathbf{Q}$ and the transition graph. Compute the new distribution π_n and study the existence of a steady-state distribution.

   We have

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

We show by induction that $\pi_n = (\frac{1}{4} + \frac{1}{2^{n+1}}, \frac{1}{4}, \frac{1}{4} - \frac{1}{2^{n+1}}, \frac{1}{4})$, hence the limit exists and is equal to $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, which is the steady-state distribution.

Exercise 2. From the following processes, which are Markov chains? For those that are, supply the transition matrix. For those that are not, give a Markov chain that is equivalent to this process, if there exists one.

1. A dice is rolled repeatedly. Let X_n be the largest number shown up to the nth roll.

   Denoting Y_n the outcome of the nth throw, $X_{n+1} = \max(X_n, Y_{n+1})$ so that

$$p_{ij} = \begin{cases} 0 & \text{if } j < i\,, \\ \frac{1}{6}i & \text{if } j = i\,, \\ \frac{1}{6} & \text{if } j > i\,. \end{cases}$$

2. A dice is rolled repeatedly. At time r, let C_r be the time since the most recent six.

   The evolution of C is given by

$$C_{r+1} = \begin{cases} 0 & \text{if the dice shows 6,} \\ C_r + 1 & \text{otherwise.} \end{cases}$$

   hence C is Markovian with

$$p_{ij} = \begin{cases} \frac{1}{6} & \text{if } j = 0\,, \\ \frac{5}{6} & \text{if } j = i+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

3. A dice is rolled repeatedly. At time r, let B_r be the time until the next six.

   This time,

$$B_{r+1} = \begin{cases} B_r - 1 & \text{if } B_r > 0\,, \\ Y_r & \text{if } B_r = 0\,, \end{cases}$$

   where Y_r is a geometrically distributed random variable with parameter $\frac{1}{6}$, independent of the sequence $B_0, B_1, \ldots, B_r$. Hence B is Markovian with

$$p_{ij} = \begin{cases} 1 & \text{if } j = i - 1 \geq 0\,, \\ (\frac{5}{6})^{j-1}\frac{1}{6} & \text{if } i = 0\,. \end{cases}$$

4. Consider a discrete event system $X_0, X_1, X_2, \ldots$ with state space S. The process is governed by two matrices $\mathbf{P}$ and $\mathbf{Q}$. If k is even, the values $\mathbf{P}[i, j]$ give the probability of going from state i to state j on the step from X_k to X_{k+1}. Likewise, if k is odd, the values $\mathbf{Q}[i, j]$ give the probability of going from state i to state j on the step from X_k to X_{k+1}.

   Not a Markov chain as the probabilities of transitions switch from one move to the other. However, encoding the parity in the state space, by doubling the number of states, makes it possible to design an equivalent Markov chain.

5. Let (X_n) be a Markov chain. Consider the process $(X_n, X_{n+1})_{n\geq 0}$.

   With $Y_n = (X_n, X_{n+1})$,

$$\begin{aligned} \mathbf{Pr}\big(Y_{n+1} = (k, \ell) \mid Y_0 = (i_0, i_1), \ldots, Y_n = (i_n, k)\big) &= \mathbf{Pr}\big(X_{n+1} = \ell \mid X_n = k\big) \\ &= \mathbf{Pr}\big(Y_{n+1} = (k, \ell) \mid Y_n = (i_n, k)\big)\,. \end{aligned}$$

   Hence, Y is Markovian with transition matrix $\mathbf{Q}$ given by

$$q_{(j,k),(k,\ell)} = p_{k,\ell}$$

   and all other coefficients are 0, where $\mathbf{P}$ is the transition matrix of X.

Exercise 3. Consider a sequence of independent, fair gambling games between two players. In each round, a player wins a coin with probability 1/2 or loses a coin with probability 1/2. The state of the system at time n is the number of coins won by player 1. The initial state is 0. Assume that there are numbers ℓ_1 and ℓ_2 such that player i cannot lose more than ℓ_i coins, and thus the game ends when it reaches one of the two states $-\ell_1$ or ℓ_2. At this point, one of the gamblers is ruined.

1. Describe the Markov chain associated with this game and find the probability that player 1 wins ℓ_2 coins before losing ℓ_1 coins.

   To each of the two states $-\ell_1$ and ℓ_2, we add a single outgoing transition that goes back to the same state. Let denote as q_ℓ the probability that starting with budget ℓ, the player 1 wins ℓ_2 coins:

$$\begin{aligned} q_{\ell_2} &= 1 \\ q_{-\ell_1} &= 0 \\ q_i &= \frac{1}{2}q_{i-1} + \frac{1}{2}q_{i+1} && \text{for } -\ell_1 < i < \ell_2 \end{aligned}$$

   For every $-\ell_1 < i < \ell_2$, we have $q_{i+1} - q_i = q_i - q_{i-1}$ and since $q_{-\ell_1} = 0$, we get $q_{i+1} - q_i = q_{-\ell_1+1}$ for every i. Summing up, we have

$$q_{i+1} - q_{-\ell_1+1} = \sum_{j=-\ell_1+1}^{i} (q_{j+1} - q_j) = (i + \ell_1)q_{-\ell_1+1}$$

   Finally, we obtain $q_{i+1} = (i + 1 + \ell_1)q_{-\ell_1+1}$. For $i = \ell_2 - 1$, this becomes $1 = q_{\ell_2} = (\ell_1 + \ell_2)q_{-\ell_1+1}$. Transposing in previous formula, we get $q_{i+1} = \frac{i+1+\ell_1}{\ell_1+\ell_2}$. Finally, for $i = -1$, we obtain $q_0 = \frac{\ell_1}{\ell_1+\ell_2}$, the probability we were searching for.

2. Classify the states and find again the probability that player 1 wins ℓ_2 coins before losing ℓ_1 coins, using this classification.

   The two states $-\ell_1$ and ℓ_2 are positive recurrent, since they are absorbing. All other states are transient, since there is a nonzero probability of moving from each of these states to either state $-\ell_1$ or state ℓ_2. For every $-\ell_1 < i < \ell_2$, state i is transient and so $\lim_{n\to\infty} p^n_{0,i} = 0$. Let q be the probability that the game ends with player 1 winning ℓ_2 coins. By definition, $\lim_{n\to\infty} p^n_{0,\ell_2} = q$. Then $1-q$ is the probability that the game ends in state $-\ell_1$. Since each round of the gambling game is fair, the expected gain of player 1 in each step is 0. Let S_n be the gain of player 1 after n steps. Then $\mathbf{E}(S_n) = 0$ for any n by induction. Thus,

$$0 = \mathbf{E}(S_n) = \sum_{i=-\ell_1}^{\ell_2} i p^n_{0,i}$$

   and

$$0 = \lim_{n\to\infty} \mathbf{E}(S_n) = \ell_2 q - \ell_1(1 - q)$$

   leading to $q = \frac{\ell_1}{\ell_1+\ell_2}$. That is, the probability of winning (or losing) is proportional to the amount of money a player is willing to lose (or win).

3. Solve the problem in the case of an unfair game with a probability $p > 1/2$ for player 1 to lose one coin on each round.

   It is not difficult to see that for every n, denoting $c = \frac{p}{1-p}$, $\mathbf{E}(c^{S_{n+1}}) = \mathbf{E}(c^{S_n})$. Hence, $\lim_{n\to\infty} \mathbf{E}(c^{S_n}) = \mathbf{E}(c^{S_0}) = 1$. This shows that $c^{\ell_2}q + c^{-\ell_1}(1 - q) = 1$ and finally the probability for player 1 to win is $\frac{1-c^{-\ell_1}}{c^{\ell_2}-c^{-\ell_1}}$.

Exercise 4. On tire à pile ou face un nombre infini de fois avec une pièce qui tombe sur pile avec probabilité $p \in]0, 1[$. Soit $k \in \mathbb{N} \setminus \{0\}$. Pour tout $n \in \mathbb{N}$, soit X_n le nombre modulo k de faces obtenus après n tirages indépendants.

1. (a) Modéliser ce processus avec une chaîne de Markov finie, à temps discret, homogène. Plus précisément, représenter la chaîne de Markov graphiquement par un graphe pondéré, puis écrire formellement sa matrice de transition P, où $P[i, j]$ dénotera l'entrée à la ligne i et colonne j. (Pour simplifier, on pourra nommer 0 le premier état et les premières ligne/colonne.)
   (b) Cette chaîne est-elle irréductible ? Apériodique ?
   (c) Montrer qu'il existe un vecteur stochastique π_∞ tel que $\pi_\infty P = \pi_\infty$.
   (d) Que peut-on dire de la transposée de P ?

2. Soit π_0 une distribution initiale pour cette chaîne de Markov. On définit par récurrence $\pi_{n+1} := \pi_n P$ pour tout $n \in \mathbb{N}$. Pour tout $n \in \mathbb{N}$, on note $\max \pi_n := \max_i \pi_n[i]$ l'entrée maximale de π_n. De même, on note $\min \pi_n := \min_i \pi_n[i]$.
   (a) Montrer que, pour tout $n \in \mathbb{N}$,
$$\max \pi_{n+1} \leq \max \pi_n \tag{1}$$
$$\min \pi_{n+1} \geq \min \pi_n \tag{2}$$
   (b) Pour tout $n \in \mathbb{N}$, soit $v_n := \max \pi_n - \min \pi_n$. Que peut-on dire de la suite $(v_n)_{n\in\mathbb{N}}$?
   (c) Soit $\alpha := (\min(p, 1-p))^k$. Montrer que pour tout i, j on a $\alpha \leq P^k[i, j]$.
   (d) En déduire que, pour tout $n \in \mathbb{N}$,
$$\max \pi_{n+k} \leq \alpha \min \pi_n + (1-\alpha) \max \pi_n \tag{3}$$
$$\min \pi_{n+k} \geq \alpha \max \pi_n + (1-\alpha) \min \pi_n \tag{4}$$
   (e) En déduire une propriété de la suite $(v_{kn})_{n\in\mathbb{N}}$.
   (f) En déduire une propriété de la suite $(\pi_n)_{n\in\mathbb{N}}$.

1. (a) $P[k-1, 0] = 1-p$ et pour tout $0 \leq i < k-1$ on a $P[i, i] = p$ et $P[i, i+1] = 1-p$.
   (b) Irréductible et apériodique : pour tout i, j il existe des chemins de i à j de longueur k et $k+1$.
   (c) $\pi_\infty := (\frac{1}{k}, \ldots, \frac{1}{k})$ convient.
   (d) La transposée de P est aussi stochastique.
2. (a) $(vP)[k-1] = \sum_{i=0}^{k-1} v[i]P[i, k-1] = v[0](1-p) + v[k-1]p \in [\min v, \max v]$, et pour $j < k-1$ on a $(vP)[j] = \sum_{i=0}^{k-1} v[i]P[i, j] = v[j]p + v[j+1](1-p) \in [\min v, \max v]$.
   (b) La suite $(v_n)_{n\in\mathbb{N}}$ est décroissante positive, donc elle converge.
   (c) Pour tout i, j, il existe un chemin de i à j de longueur k. Chaque arête du chemin est étiquetée par p ou $1-p$. La probabilité de prendre ce chemin est donc supérieure à $(\min(p, 1-p))^k$, donc $\alpha \leq P^k[i, j]$.
   (d) Pour tout j on a
$$\begin{aligned}
(\pi_n P^k)[j] &= \sum_{i=0}^{k-1} \pi_n[i]P^k[i, j] \\
&= \pi_n[i_0]P^k[i_0, j] + \sum_{i\neq i_0} \pi_n[i]P^k[i, j] \text{ où } \pi_n[i_0] := \min \pi_n \\
&\leq (\min \pi_n)P^k[i_0, j] + (\max \pi_n) \sum_{i\neq i_0} P^k[i, j] \\
&= (\min \pi_n)P^k[i_0, j] + \max \pi_n(1 - P^k[i_0, j]) \\
&\leq \min \pi_n \alpha + \max \pi_n(1-\alpha)
\end{aligned}$$
   (e) Donc
$$\max \pi_{n+k} \leq \alpha \min \pi_n + (1-\alpha) \max \pi_n$$
$$-\min \pi_{n+k} \leq -\alpha \max \pi_n + -(1-\alpha) \min \pi_n$$

En additionnant et en remplaçant n par kn on trouve $v_{k(n+1)} \leq (1-2\alpha)v_{kn}$, donc $v_{kn} \leq (1-2\alpha)^n v_0$. Donc la suite des $v_{kn} \geq 0$ converge vers 0, car $(1-2\alpha) < 1$. (En effet, $0 < \alpha$ car $p \in]0,1[$.)

(f) Tous les π_n sont stochastiques, donc $(\pi_n)_{n\in\mathbb{N}}$ converge vers π_∞ : soit $\epsilon > 0$, il existe $n_0 \in \mathbb{N}$ tel que pour tout $n \geq n_0$, on a $\max \pi_n - \min \pi_n \leq \epsilon$. Or $\sum_i \pi_n[i] = 1$, donc

$$\begin{aligned} k \max \pi_n &= 1 + \sum_i \max \pi_n - \pi_n[i] k \\ &\leq 1 + k\epsilon \end{aligned}$$

De même $\leq 1 - k\epsilon \leq k \min \pi_n$, donc pour tout i on a $\frac{1}{k} - \epsilon \leq \pi_n[i] \leq \frac{1}{k} + \epsilon$. Cela montre que $\pi_n[i]$ converge vers $\frac{1}{k}$ pour tout i.