Probabilistic Aspects of Computer Science: TD6
Probabilistic Automata

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Exercise 1. We consider the probabilistic automaton $A$ over alphabet $A = \{a, b\}$ depicted below:

![Diagram of a probabilistic automaton]

1. Describe the probability associated with word $ab^n$ for every natural number $n$.
2. What are the possible probabilities that can be associated to a word by $A$?
3. Determine the (finite) class $L = \{L_{>\theta}(A) \mid \theta \in [0, 1]\}$.

Exercise 2. Prove or disprove that the following languages are stochastic languages.

1. $\{w \in \{a, b\}^+ \mid |w|_a = |w|_b\}$, $\{w \in \{a, b, c\}^+ \mid |w|_a = |w|_b = |w|_c\}$ and $\{a^nb^n \mid n > 0\}$
2. $\{a^nb^m \mid n \neq m\}$
3. $\{amba^m ba^m b \cdots a^mba^n \mid \exists j \leq k \ m_1 + m_2 + \cdots + m_j = m\}$
4. language of palindromes over $\{a, b\}$, i.e., words $w$ such that $w = \overline{w}$
5. $\{a^nb^m a^nb^m \mid n, m \geq 1\}$

Exercise 3 (Isolated cutpoints). Let $A = (Q, A, \{P_a\}_{a \in A}, \pi_0, F)$ be a probabilistic automaton and let $\theta \in [0, 1]$. We say that $\theta$ is an isolated cut point of $A$ if there is $\delta > 0$ such that for all $w \in A^*$, we have $|\Pr_A(w) - \theta| \geq \delta$. In the following, we will consider that $Q = \{1, \ldots, n\}$ with $1$ the unique initial state (i.e., $\pi_0(1) = 1$).

1. Let $L = L_{>\theta}(A)$. We consider the Myhill-Nerode congruence $\equiv_L \subseteq A^* \times A^*$ given as follows: for $u, v \in A^*$,

   $$u \equiv_L v \text{ iff. } \forall w \in A^* \ (uw \in L \iff vw \in L)$$

   As we know, $\equiv_L$ has finite index (i.e., admits a finite set of equivalence classes) iff. $L$ is regular. We also define for $u \in A^*$, $\xi^u = (P_{u[1,1]}, \ldots, P_{u[1,n]})$. Show that if $\theta$ is an isolated cut point, $u \not\equiv_L v \implies \|\xi^u - \xi^v\|_1 \geq 4\delta$.

2. Deduce that if $\theta$ is an isolated cut point of $A$, the language $L_{>\theta}(A)$ is regular.

3. We assume that the following variant of PCP is undecidable: Given an alphabet $A$ and two morphisms $\varphi_1, \varphi_2 : A \rightarrow \{0, 1\}^*$, is $\{\varphi_1(w) \land \varphi_2(w) \mid w \in A^*\}$ finite? Let $\varphi_1$ and $\varphi_2$ an instance of this problem. Show that there exists $\delta > 0$ such that for all $w \in A^+$, $|\bin(\varphi_1(w)) - \bin(\varphi_2(w))| \geq \delta$. Deduce that the following problem is undecidable: given a probabilistic automaton $A$ and $\theta \in (0, 1)$, is $\theta$ an isolated cut point of $A$?

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1 We denote $u \land v$ the longest common suffix of the words $u$ and $v$. You may find the proof of undecidability in V. Blondel and V. Canterini: Undecidable Problems for Probabilistic Automata of Fixed Dimension in Theory of Computing Systems, 2001.

2 We denote by $\bin(0.w)$ the unique rational number between 0 and 1 having as binary decomposition $0.w$, where $a$ is interpreted as 0 and $b$ as 1.