Probabilistic Aspects of Computer Science: TD3
Random walks and applications

Adrien Koutsos

September 27, 2016

Exercise 1 (Cover time)**.** Let $G = (V, E)$ be a finite, undirected, and connected graph. A random walk on G is a Markov chain defined by the sequence of moves of a particle between vertices of G. In this process, the place of the particle at a given time step is the state of the system. If the particle is at vertex i and if i has $d(i)$ outgoing edges, then the probability that the particle follows the edge $\{i, j\}$ and moves to a neighbor j is $1/d(i)$.

1. Show that a random walk on an undirected graph G is aperiodic if and only if G is not bipartite.
2. In the rest of the exercise, we assume that G is not bipartite. Show that a random walk on G converges to a steady-state distribution π, where $\pi_v = \frac{d(v)}{2|E|}$.
3. We consider a new Markov chain defined on the edges of G. The current state is defined to be the pair composed of the edge most recently traversed in the random walk, together with the direction of this traversal: the state space is hence the set of directed edges. There are $2|E|$ states in this new Markov chain, and its transition matrix $\mathbf{Q}$ is given by:

$$\mathbf{Q}_{(u,v),(v,w)} = \begin{cases} \frac{1}{d(v)} & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise} \end{cases}$$

   Compute its steady-state distribution.
4. We denote $\mu_{v,u}$ the expected number of steps to reach u from v. Show that if $\{u, v\} \in E$, then $\mu_{u,v} + \mu_{v,u} \leq 2|E|$ *(Hint: use the result of the previous question).*
5. The *cover time* of G is defined as the maximum over all vertices $v \in V$ of the expected time to visit all of the nodes in the graph by a random walk starting from v. Show that the cover time of G is bounded above by $2|E|(|V| - 1)$.
6. As an application, suppose we are given an undirected graph $G = (V, E)$ and two vertices s and t in G, and we want to determine whether there is a path connecting s and t. For simplicity, assume that the graph G has no bipartite connected components. By standard deterministic search algorithms, we can easily solve the problem in linear time, using $\Omega(n)$ space. Show that the following algorithm returns the correct answer with probability $1/2$, and it only errs by returning that there is no path from s to t when there is such a path. What is the time and space complexities of this algorithm? (*Hint: you may use the Markov's inequality, which says for a random variable X and $a > 0$ that* $\mathbf{Pr}(|X| \geq a) \leq \frac{\mathbf{E}(|X|)}{a}$.)

s-t **Connectivity algorithm**
1. Start a random walk from s.
2. If the walk reached t within $2|V|^3$ steps, return that there is a path. Otherwise, return that there is no path.

Exercise 2 (Lumpability and Couplings)**.** The goal of this exercice is to introduce the notions of lumpability of a Markov Chain, and of probabilistic couplings.

1. **Lumpability:** consider a Markov Chain $(X_n)_{n\in\mathbb{N}}$ with states $\mathcal{S}$ and transition probabilities given by P. Assume that we have a decomposition of $\mathcal{S}$ into (non-empty disjunct sets) $\mathcal{S}_1, \dots, \mathcal{S}_l$ such that the following property (called the strong lumpability property) holds:

$$\forall i, j, \forall s, s' \in \mathcal{S}_i,\ \mathbf{Pr}(X_{n+1} \in \mathcal{S}_j \mid X_n = s) = \mathbf{Pr}(X_{n+1} \in \mathcal{S}_j \mid X_n = s')$$

Let $\mathcal{C}^{\mathsf{ag}}$ be the Markov Chain with states $\{\mathcal{S}_1, \dots, \mathcal{S}_l\}$ whose initial distribution and transition propabilities are:

$$\forall i,\ \pi^{(0)}_{\mathsf{ag}}[i] = \sum_{s\in\mathcal{S}_i} \pi^{(0)}[s]$$

$$P^{\mathsf{ag}}[i][j] = \sum_{s'\in\mathcal{S}_j} P[s][s'] \qquad \text{(for some } s \in \mathcal{S}_i\text{)}$$

Show that for all n, $\pi^{(n)}_{\mathsf{ag}}[i] = \sum_{s\in\mathcal{S}_i} \pi^{(n)}[s]$.

2. **Couplings:** consider two Markov Chain $\mathcal{M}_l, \mathcal{M}_r$ with the same set of states $\mathcal{S}$ and transition probabilities given by P_l and P_r. Let $\mathcal{R}$ be a relation on $\mathcal{S} \times \mathcal{S}$, and suppose that $\{x \mid \exists y,\ (x, y) \in \mathcal{R}\} = \mathcal{S}$ and $\{y \mid \exists x,\ (x, y) \in \mathcal{R}\} = \mathcal{S}$. A $\mathcal{R}$-coupling of $\mathcal{M}$ is a Markov Chain $\mathcal{M}_c$ with state space $\mathcal{R}$ and transition probabilities P_c such that:

   - *Left Marginal:* For all $(s_0, s'_0) \in \mathcal{R}, s_1 \in \mathcal{S}$ we have:

   $$\sum_{\{s'_1 \mid (s_1, s'_1)\in\mathcal{R}\}} P_c[(s_0, s'_0)][(s_1, s'_1)] = P_l[s_0][s_1]$$

   - *Right Marginal:* For all $(s_0, s'_0) \in \mathcal{R}, s'_1 \in \mathcal{S}$ we have:

   $$\sum_{\{s_1 \mid (s_1, s'_1)\in\mathcal{R}\}} P_c[(s_0, s'_0)][(s_1, s'_1)] = P_r[s'_0][s'_1]$$

   Show the following properties of couplings:

   (a) Show that if $(X_n, Y_n)_{n\in\mathbb{N}}$ is a stochastic process following $\mathcal{M}_c$, then the left projection $(X_n)_{n\in\mathbb{N}}$ (resp. right projection $(Y_n)_{n\in\mathbb{N}}$) is a Markov Chain with state space $\mathcal{S}$ and transition probabilities P_l (resp. P_r).

   (b) Show that any Markov Chain admits a (trivial) coupling.

3. **Reachability Couplings:** We focus on reachability coupling: let $\mathcal{T} \subseteq \mathcal{S}$ be an absorbing subset in $\mathcal{M}_l$ and $\mathcal{M}_r$ (i.e. $\forall s \in \mathcal{T}, s' \in \mathcal{S},\ P_l[s][s'] > 0 \Rightarrow s' \in \mathcal{T}$ and $P_r[s][s'] > 0 \Rightarrow s' \in \mathcal{T}$). A $\mathcal{R}$-coupling is a reachability coupling for $\mathcal{T}$ if $\mathcal{R}$ is of the form:

$$\mathcal{R} = \{(s, s') \in \mathcal{S}^2 \mid s = s' \vee s \in \mathcal{T}\}$$

   (a) Show that in that case we have that for all n:

   $$\mathbf{Pr}(X_n \notin \mathcal{T}) \leq \mathbf{Pr}(Y_n \notin \mathcal{T})$$

   (b) Let T^{reach}_l (resp. T^{reach}_r) be the time taken to reach $\mathcal{T}$ in $\mathcal{M}_l$ (resp. $\mathcal{M}_r$). Deduce from the previous question the following relation on the expected time to reach $\mathcal{T}$ in the left and right Markov Chains:

   $$\mathbf{E}(T^{\mathsf{reach}}_l) \leq \mathbf{E}(T^{\mathsf{reach}}_r)$$

4. **Application to a coloring problem:** A coloring of a graph is an assignment of a color to each of its vertices. A graph is k-colorable if there is a coloring of the graph with k colors such that no two adjacent vertices have the same color. An almost 2-coloring of a graph G is a coloring of G with two colors such that no triangle is monochromatic. Let G be a 3-colorable graph.

(a) Show that there exists a coloring of the graph with two colors such that no triangle is monochromatic.

(b) Consider the following algorithm looking for an almost 2-coloring of the vertices of G. The algorithm begins with an arbitrary coloring c of G with 2 colors. While there are any monochromatic triangles in G, the algorithm chooses deterministically one such triangle and changes the color of a randomly chosen vertex of that triangle. Our goal is to derive an upper bound on the expected number of such recoloring steps before the algorithm finds an almost 2-coloring of G.

Let $\mathcal{C}$ be any 3-coloring of the graph (for example with colors red, blue and green). Suppose that we use colors blue and green in our colorings with two colors. Let $\mathcal{T}$ be the set of almost 2-coloring of G and $\mathcal{A}$ be the set of coloring (using only green and blue) of G that are not almost 2-coloring. For all coloring $c \in \mathcal{A}$, we let $\mathsf{mono}(c)$ be a monochromatic triangle in G with coloring c. Describe the algorithm as a Markov Chain $\mathcal{M}_l$ with state space $\mathcal{S} = \mathcal{A} \cup \mathcal{T}$ (we will use P to denote its transition matrix). Since the algorithm terminates when it reaches a coloring $c \in \mathcal{T}$, we will choose that $\forall c \in \mathcal{T}, P[c][c] = 1$ (therefore $\mathcal{T}$ is absorbing).

(c) Let $k(c)$ be the number of vertices which have the same colors in $\mathcal{C}$ and the coloring c. We let $k_{min} = \min_{c\in\mathcal{A}}(k(c))$ and $k_{max} = \max_{c\in\mathcal{A}}(k(c))$. Moreover for all $k \in \{k_{min}, \ldots, k_{max}\}$, we let $\mathcal{A}^k = \{c \in \mathcal{A} \mid k(c) = k\}$. Show that for all n:

$$\forall c \in A^k, P[c][\mathcal{A}^k] = \mathbf{Pr}_l(X_{n+1} \in \mathcal{A}^k \mid X_n = c) \in \{0; 1/3\} \qquad (k_{min} \le k \le k_{max})$$
$$\forall c \in A^k, P[c][\mathcal{A}^{k+1}] = \mathbf{Pr}_l(X_{n+1} \in \mathcal{A}^{k+1} \mid X_n = c) \in \{0; 1/3\} \qquad (k_{min} \le k < k_{max})$$
$$\forall c \in A^k, P[c][\mathcal{A}^{k-1}] = \mathbf{Pr}_l(X_{n+1} \in \mathcal{A}^{k-1} \mid X_n = c) \in \{0; 1/3\} \qquad (k_{min} < k \le k_{max})$$

Moreover show that:

$$P[c][\mathcal{T}] + \sum_{l\in\{k-1,k,k+1\}} P[c][\mathcal{A}^l] = 1 \qquad (c \in \mathcal{A}^k \text{ and } k_{min} < k < k_{max})$$
$$P[c][\mathcal{T}] + \sum_{l\in\{k_{min},k_{min}+1\}} P[c][\mathcal{A}^l] = 1 \qquad (c \in \mathcal{A}^{k_{min}})$$
$$P[c][\mathcal{T}] + \sum_{l\in\{k_{max}-1,k_{max}\}} P[c][\mathcal{A}^l] = 1 \qquad (c \in \mathcal{A}^{k_{max}})$$

Deduce that:

$$\forall c \in A^{k_{max}}, \mathbf{Pr}_l(X_{n+1} \in \mathcal{T} \mid X_n = c) \ge 1/3$$
$$\forall c \in A^{k_{min}}, \mathbf{Pr}_l(X_{n+1} \in \mathcal{T} \mid X_n = c) \ge 1/3$$

(d) Assume that for all $k \in \{k_{min}, \ldots, k_{max}\}$, there exists an element $a^k \in \mathcal{A}^k$ (we could easily remove this assumption by adding "dummy" elements). Construct a Markov Chain $\mathcal{M}_r$ with the same set of states $\mathcal{S}$ but transition matrix P' such that all the transitions have probability 1/3, i.e. for all n:

$$\forall c \in A^k, \mathbf{Pr}_r(X_{n+1} \in \mathcal{A}^k \mid X_n = c) = 1/3 \qquad (k_{min} \le k \le k_{max})$$
$$\forall c \in A^k, \mathbf{Pr}_r(X_{n+1} \in \mathcal{A}^{k+1} \mid X_n = c) = 1/3 \qquad (k_{min} \le k < k_{max})$$
$$\forall c \in A^k, \mathbf{Pr}_r(X_{n+1} \in \mathcal{A}^{k-1} \mid X_n = c) = 1/3 \qquad (k_{min} < k \le k_{max})$$

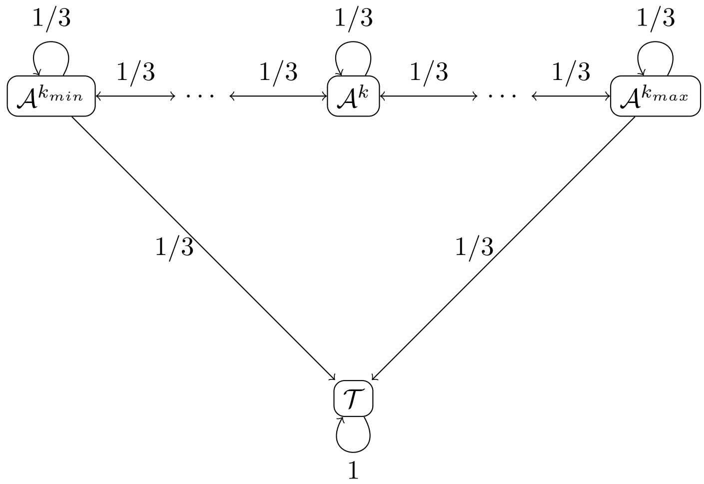
And:

$$\forall c \in A^{k_{max}}, \mathbf{Pr}_r(X_{n+1} \in \mathcal{T} \mid X_n = c) = 1/3$$
$$\forall c \in A^{k_{min}}, \mathbf{Pr}_r(X_{n+1} \in \mathcal{T} \mid X_n = c) = 1/3$$

And show that there exists a reachability coupling between $\mathcal{M}_l$ and $\mathcal{M}_r$ with target set $\mathcal{T}$.

Hint: Follow the behavior of $\mathcal{M}_l$ as long as it stays in $\mathcal{A}$, but change the behavior (by using the elements $\{a^k \mid k_{min} \le k \le k_{max}\}$) if the left chain goes "too early" in $\mathcal{T}$.

(e) Show that the Markov Chain $\mathcal{M}_r$ satisfies the strong lumpability property for the decomposition $\{\mathcal{A}^k \mid k_{min} \leq j \leq k_{max}\} \cup \{\mathcal{T}\}$, and show that the resulting lumped chain $\mathcal{M}_r^{\mathsf{lump}}$ is of the form:



(f) Deduce that the expected time $\mathsf{T}_r^{\mathsf{reach}}$ to reach $\mathcal{T}$ in $\mathcal{M}_r$ is upper bounded by $\mathcal{O}(n^2)$ (where n is the number of vertices of G).

Hint: use the upper bound on the cover time of a random walk in a graph shown in Exercice 1.

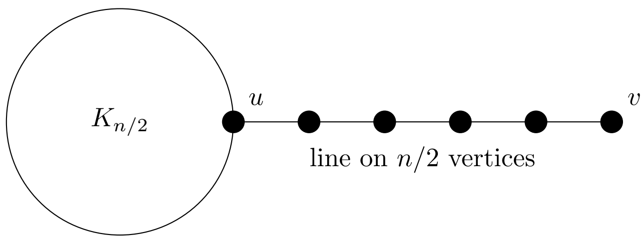
Deduce from this an upper bound on the expected time $\mathsf{T}_l^{\mathsf{reach}}$ to reach $\mathcal{T}$ in $\mathcal{M}_l$, using the result on reachability couplings shown before.

Exercise 3 (Cat and mouse)**.** A cat and a mouse each independently take a random walk on a connected, undirected, non-bipartite graph G, with n vertices and m edges. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Show an upper bound of $O((nm)^2)$ on the expected time before the cat eats the mouse. What is a good strategy for the cat to eat quickly the mouse?

Exercise 4 (Computation of cover times)**.**

1. What is the cover time of a line when we start on an end node of the line?
2. What is the cover time of a complete graph?

The lollipop graph on n vertices is a clique on $n/2$ vertices connected with a line on $n/2$ vertices as shown below:



The node u is a part of both the clique and the line. Let v denote the other end of the line.

3. Show that the cover time of a random walk starting at v is $\Theta(n^2)$.
4. Show that the cover time of a random walk starting at u is $\Theta(n^3)$.

Exercise 5. For the following random walks, give the classification of the states (transient, null recurrent, or positive recurrent) and tell whether they admit a steady-state distribution.

1. the random walk over $\mathbb{Z}$?

2. the random walk on the 2-dimensional integer lattice, where each point has four neighbors (up, down, left, and right)?
3. the random walk on the 3-dimensional integer lattice?