Orders.

Exercice 1 :

Let E be a set with a partial order $\leq$. Recall that an *antichain* is a subset of E in which all the elements are incomparable.

1. We consider $\mathbb{N}^2$ with the product order $((a,b) \leq (x,y) \iff a \leq x \wedge b \leq y)$
   (a) Show an antichain of cardinality n, for any $n > 1$.
   (b) Can we find an infinite antichain ?
2. Show that the set Σ^* with the sub-string order ($w_1 < w_2$ iff $\exists u, v \in \Sigma^*$ s.t. $w_2 = uw_1v$), has an infinite chain.

Exercice 2 :

Given $A, B \in \mathcal{P}([n])$ we say that $A < B$ iff $A \subset B$.

1. Assume that $n > 1$. Show that :
$$1 < \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \cdots > \binom{n}{n-2} > \binom{n}{n-1} > 1$$
2. For $k \in \{1, ..., \frac{n}{2}\}$ find an antichain of cardinality $\binom{n}{k}$ in $\mathcal{P}([n])$.
3. Let A be an anti chain in $\mathcal{P}([n])$. For k in $[\![0, n]\!]$, we denote by a_k the number of sets of cardinality k in A. We will now show the Lubell-Yamamoto-Meshalkin inequality :
$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1$$
   (a) Demonstrate that there are exactly $n!$ Strictly increasing chains in $\mathcal{P}([n])$, of the form $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$.
   (b) Let S be a subset of X of cardinality s. Show that there are exactly $s!(n-s)!$ strictly increasing chains in $\mathcal{P}([n])$, of the form $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$, where $X_s = S$.
   (c) Let $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_r$ a strictly increasing chains in $\mathcal{P}([n])$. Then there is at most one X_i in A(the antichain). By partitioning all the strictly increasing sequences $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$, according to their possible intersection with A, demonstrate the Lubell-Yamamoto-Meshalkin inequality.
4. Deduce the maximal cardinality of an antichain in $\mathcal{P}([n])$.

Exercice 3 :

Show 2 of the following :

Let $(R_i)_{i \in I}$ a family of binary relationships on the set E. Let $R = \cap_{i \in I} R_i$. In other words, xRy if xR_iy for all $i \in I$.

- If one of the R_i's is irreflexive/asymmetric/antisymmetric, then R is too.
- If all R_i's are reflexive/symmetric/transitives, then R is too
- The intersection does not preserve totality or trichotomy.

Exercice 4 :

Show 2 of the following :

Let R be a relation on the set E, and $F \subseteq E$. If R is reflexive / symmetric / transitive / total / antisymmetric / irreflexive / asymmetric / trichotomous, then R_F is the same.

Exercice 5 :
Show 2 of the following :

If R is reflexive / symmetric / transitive / total / antisymmetric / irreflexive / asymmetric / trichotomous, Then R^{-1} is the same.

Exercice 6 :
Show the following propositions :

- If $(E, \leq)$ is a finite ordered set and $x \in E$. There is a maximal element y in E s.t. $x \leq y$.
- If any finite part of an ordered set has a greatest element, then it is a total order.

Exercice 7 :
Show that the lexicographic product of (total) orders is a (total) order.
For all $x, y \in \prod_{i \in I}(E_i, \leq_i)$, $x \leq_{lex} y$ iff given $k := \min\{j \in I \,|\, x_j \neq y_j\}$ we have $x_k \leq_k y_k$.

Exercice 8 :
Let E be a set with a prtial order denoted by $\preccurlyeq$. We say that $\preccurlyeq$ is a well quasi-order(wqo) if from any sequence of elements of E, we can extract an infinite monotone increasing sequence. I.e. $\forall (x_i)_{i \in \mathbb{N}} \in E^{\mathbb{N}}$, there exists an increasing sub-sequence of indexes : $i_0 < i_1 < \cdots < i_n < \cdots$ for which the sequence $(x_{i_n})_{n \in \mathbb{N}}$ is increasing : $x_{i_0} \preccurlyeq x_{i_1} \preccurlyeq \cdots \preccurlyeq x_{i_n} \preccurlyeq \cdots$.

1. Show that if the order $\preccurlyeq$ is total, then it is wqo iff any non-empty subset of E has a least element.
2. Give an example of a total ordered set which is not wqo.
3. Show that the following are equivalent :
   - (*def1*) The ordered set $(E, \preccurlyeq)$, is wqo.
   - (*def2*) For any sequence $(x_i)_{i \in \mathbb{N}}$, we can find $i < j$ s.t. $x_i \preccurlyeq x_j$.
   - (*def3*) (i) There is no infinite sequence strictly decreasing in E,
     (ii) There is no infinite antichain.
4. Let $E, \preccurlyeq$ be an ordered set, we call it well founded if there is no infinite decreasing sequence. Assume that E is countable and show that the order is wqo iff the set of all antichains is countable.
5. Dickson's Lemma : Let $(E_1, \preccurlyeq_1)$ and $(E_2, \preccurlyeq_2)$ be a wqo set. Show that $(\preccurlyeq_1, \preccurlyeq_2)$ is a wqo on the product $E_1 \times E_2$.
6. Higman's Lemma : Let $\preccurlyeq$ be a wqo on Σ. Define a relationship on Σ^* as follows :

$$a_1 \ldots a_m \leq_{sw} b_1 b_2 \ldots b_n \Leftrightarrow \begin{cases} \exists 1 \leq i_1 < i_2 < \cdots < i_m \leq n \\ a_1 \preccurlyeq b_{i_1} \wedge a_2 \preccurlyeq b_{i_2} \cdots a_m \preccurlyeq b_{i_m} \end{cases}$$

   (a) Show that $\leq_{sw}$ is an order
   (b) Show that $\leq_{sw}$ is wqo.
7. Let $E, \leq$ be an ordered set, and $F \subset E$ s.t. : $\forall y \in E$, if there exists $x \in F$ s.t $x \preccurlyeq y$, then $y \in F$ (we say that the set F is *upward closed* or *upper*).
   (a) Let $E, \leq$ be wqo. Show that a that any increasing sequence of upward closed sets is stationary, i.e. $F_1 \subseteq F_2 \subseteq \cdots$ there exists i such that for any $j > i$ $F_i = F_j$.
   (b) Show that if F is an upward closed set, there exists a finte set of elements $x_1, \ldots, x_n$ in F s.t. $F = \cup_i \{y \in E, x_i \preccurlyeq y\}$.

Exercice 9 (Dilworth's theorem) :

Let $(E, \leq)$ be a finite ordered set..

Let F be a non-empty subset of E. Denote by $\text{Max}(F)$ all of its maximal elements.

1. if F is a non-empty subset of E, show that $\text{Max}(F)$ is a non-empty subset of F.
2. If F is a non-empty subset of E, show that $\text{Max}(F)$ is an antichain and that it is maximal(inclusion-wise) of the antichains in F

We now show by induction on $|E|$, **Dilworth's theorem** :

Let k be the maximal cardinality of an antichain in E. Then E is a disjoint union of k chains(a set of comparable elements).

3. Demonstrate the result when $k = 1$.
4. Suppose that $k > 1$. Let $z \in \text{Max}(E)$ and $F = E \setminus \{z\}$. Let l the maximal cardinality of an antichain in F ; by induction assumption F is a disjoint union of chains $C_1, ..., C_l$.
   (a) Give a bound on k as a function of l.
   (b) Let $\mathcal{C}$ be an antichain of E. What can we say about $\mathcal{C} \cap C_i$, for $i \in [l]$?
   (c) Let $i \in [l]$. We denote by D_i the set of elements of C_i that are in an antichain of cardinality l in F.
      i. Show that D_i isn't empty.
      Denote by y_i the maximal element of D_i.
      ii. Show that $\{y_1, ..., y_l\}$ is an antichain of F.
      iii. Assume that for every i, z is incomparable to y_i. Conclude Dilworth's theorem for this case.
      iv. Assume that there exists an i s.t. z is comperable to y_i. Let $C' = \{z\} \cup \{x \in C_i \mid x \leq y_i\}$. Show that C' is a chain and that $F \setminus C'$ does not contain an antichain of cardinality l. Conclude Dilworth's theorem.