Exercice 0 :

1. Show that $[n]$ has $n!$ permutation. ($[n] = \{1, 2, \dots, n\}$)
2. Let $n, m \in \mathcal{N}$ and $(x_i)_{i \in [\![1, mn+1]\!]}$ be a sequence of natural numbers. Show that the given sequence admits an non-decreasing sub-sequence of length $n+1$ or a non-increasing sub-sequence of length $m+1$.

Exercice 1 :

Prove the following identities using combinatorial arguments :

1. $\sum_{0 \leq 2i \leq n} \binom{n}{2i} = 2^{n-1}$ et $\sum_{0 \leq 2i+1 \leq n} \binom{n}{2i+1} = 2^{n-1}$.
2. $\sum_{i=0}^{n} i \binom{n}{i} = n2^{n-1}$.
3. $\binom{n}{l}\binom{l}{k} = \binom{n}{k}\binom{n-k}{l-k}$, for $0 \leq k \leq l \leq n$.
4. Given $m, n \in \mathcal{N}$ such that $1 \leq m \leq n$.
$$\sum_{i=m}^{n} \binom{n}{i}\binom{i}{m} = 2^{n-m}\binom{n}{m}$$
5. For all $n \geq 2$:
$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$
6. For all $n \geq 3$:
$$\sum_{k=3}^{n} k(k-1)(k-2)\binom{n}{k} = n(n-1)(n-2)2^{n-3}$$

Exercice 2 :

Let $N > 0$ be a natural number

1. Let $\mathcal{E}_2$ be the a family consisting of all subsets of size 2 of the set $[N]$. Partition the family $\mathcal{E}_2$ in a good manner in order to recover the equality :
$$\sum_{j=1}^{N-1} j = \frac{N(N-1)}{2}$$
2. By partitioning the set $[N]^3$ according to the maximal value of its items (i.e. (x, y, z) and (x', y', z') are in the same partition if $\max(x, y, z) = \max(x', y', z')$), recover a the expression $\sum_{j=1}^{N-1} j^2$ as a function of N.

Exercice 3 :

Given $n_1, \dots, n_{12}$ a familly of 12 integers. Show there exist $i \neq j$ such that $n_i - n_j$ is a multiple of 11 (i.e. $(n_i - n_j) \mod 11 = 0$).

Exercice 4 :

Show that, in a group of 6 people there always exists either a sub-group of 3 people who don't know each other, or a sub-group of 3 people who all know each other.

Exercice 5 :

A pass word is considered *valid* if it satisfies the following conditions :

- It consists of 8 characters taken from the 26 letters of the alphabet, the numbers 0 et 9, and the 7 special characters !, ?, %, #, @, &, $.
- It includes at least one letter from the alphabet.
- It includes at least one number.
- It includes at least one special character.

Determine the number of valid passwords.

Exercice 6 :

Let $m, n \in \mathcal{N}$. Denote by $s(m, n)$ the number of surjective function from the set $[m]$ to the set $[n]$.

1. What is $s(m, n)$ if $m < n$? and if $m = n$?
2. Prove the following formula using the inclusion–exclusion principle :

$$s(m,n) = n^m - n(n-1)^m + \binom{n}{2}(n-2)^m + \cdots + (-1)^k \binom{n}{k}(n-k)^m + \cdots + (-1)^n n$$

Exercice 7 (Ramsey's theorem) :

1. Show that $\forall (n_r, n_b) \in \mathbb{N}^2, \exists N \in \mathbb{N}$ such that, for any 2 (edge) coloring $\{r, b\}$ of the complete graph K_N, there exists a color $c \in \{r, b\}$ for which there is a complete sub-graph K_{n_c} which is monochromatic in the color c.
(the smallest N for which this property holds is denoted by $R(n_r, n_b)$).
2. Show that $\forall k \in \mathbb{N}, \forall (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k, \exists N \in \mathbb{N}$ such that, for any k (edge) coloring of the complete graph K_N, there exists a color $c \in [\![1, k]\!]$ for which there is a complete sub-graph K_{n_c} which is monochromatic in the color c.
(the smallest N for which this property holds is denoted by $R(n_1, \ldots, n_k)$).