Gröbner Basis and deducibility

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July 13, 2018

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Introduction to security

A rising need for security

$$\text{Confidentiality:} \quad \begin{cases} \text{Banking operations} \\ \text{Smartphones with GPS} \\ \dots \end{cases}$$

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$$\text{Authentication:} \quad \begin{cases} \text{Internet shopping} \\ \text{Private emails} \\ \dots \end{cases}$$

Cryptographic primitives

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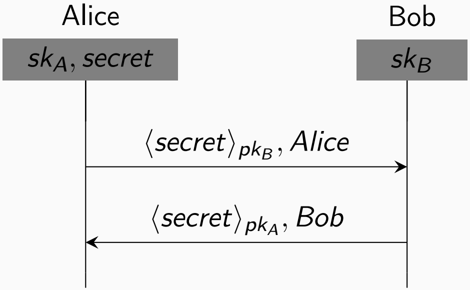
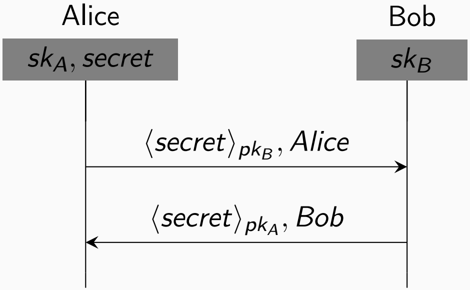
based on exponentiation in group and hardness of discrete logarithm

Protocols

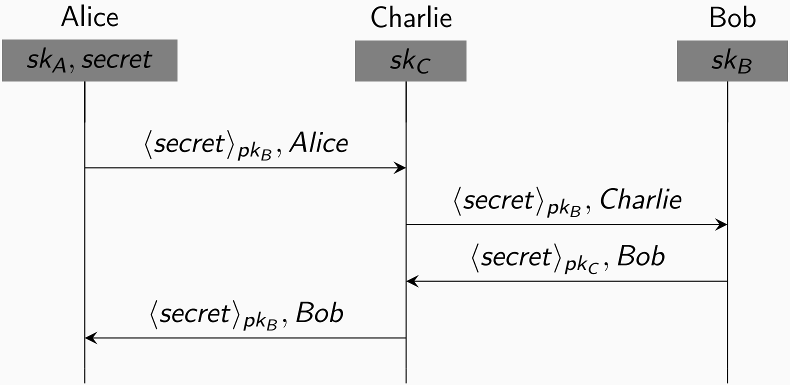

Alice
Bob
$sk_A, secret$
sk_B
$\langle secret \rangle_{pk_B}, Alice$
$\langle secret \rangle_{pk_A}, Bob$

Protocols



Is secret secret ?

Protocols



No !

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$\hookrightarrow$ We want automation

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$$g^{y^2+y} = (g^{x \times y})^{x^{-1}} \times g^{y^2}$$

Our generalized problem

$$\Gamma \models X, g^{f_1}, ..., g^{f_k} \vdash g^h \qquad \begin{cases} \Gamma \text{ axioms} \\ X \text{ public variables} \\ Y \text{ secret variables} \\ g \text{ group} \\ f_i, h \text{ polynomials over } \mathbb{K}[X, Y] \end{cases}$$

The algorithm

$$\Gamma \models X, g^{f_1}, \ldots, g^{f_k} \vdash g_t^h$$

With Gröbner Basis

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1. Characterize the attacker knowledge:

$$M = \{\sum_i e_i \times f_i | e_i \in \mathbb{K}[X]\}$$

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2. Saturate using the axioms, if $\Gamma = \{p_k \neq 0\}$:

$$M :_{\mathbb{K}[X,Y]} (p_1 ... p_n)^\infty = \{f \in \mathbb{K}[X, Y] | \exists n \in \mathbb{N}, f \times (p_1 ... p_n)^n \in M\}$$

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Gröbner Basis

(in)Formal definition

For f, g_i in $\mathbb{K}[X, Y]$, we have:

- an ordering on monomials
- if $lm(f) = qlm(g_1)$ then $red_1^X(f, g_1) = f - qg$
- $red^X(f, g_i)$ is the iteration of red_1 for all g_i

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Gröbner Basis

$$G = \{g_i\} \text{ is a GB pf } M \text{ iff } \forall h \in M, red(h, G) = 0$$

Buchberger's algorithm

For f, g_i in $\mathbb{K}[X, Y]$, we have:

$$S(f, g_1) = \frac{lppcm(f, g_1)}{lm(g)} f - \frac{lppcm(f, g_1)}{lm(f)} g$$

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Goal

If $M =< G >$, compute the GB of

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Magic trick

- compute the GB of $G \cup (1 - tp)$, with t a fresh variable
- keep only the base element not containing t.

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It is a good idea to to have general knowledge in math when doing computer science !