Composition in the BC model

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Introduction
Who am I?
A third year PhD Student, working in Paris and Nancy, supervised by:

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The goal

We want security!
We want formal proofs of security, in the computational model.
The goal

**We want security !**
We want formal proofs of security, in the computational model.

**But:**

- There is few automation;
- proofs are long and error-prone;
- there is no modularity;
- and proofs size grows w.r.t to the size of the protocol.
Our contributions

The composition framework

- Allows to split the security of an unbounded number of sessions of a compound protocol into smaller finite goals;
Our contributions

The composition framework

• Allows to split the security of an unbounded number of sessions of a compound protocol into smaller finite goals;
• allows to consider protocols with state passing and long term shared secrets;
Our contributions

The composition framework

• Allows to split the security of an unbounded number of sessions of a compound protocol into smaller finite goals;
• allows to consider protocols with state passing and long term shared secrets;
• naturally translates to the BC model, and allows for the first time to perform proofs for an unbounded number of sessions in this model.
The BC model?
A quick introduction to the BC model

A protocol

$A \xrightarrow{\text{sign}(r,skA)} B$
A quick introduction to the BC model

A protocol

\[ A \xrightarrow{\text{sign}(r,skA)} B \]

| Checks the signature

In BC Protocols are modelled with sequences of terms:

\[ \phi_0 := \text{sign}(r,skA) \]

\[ \phi_1 := \phi_0, \text{if} \left( \text{checksign}(g_0(\phi_0),pk(sk_A)) \right) \text{then} \]

\[ \langle \text{"ok"}, \text{getmess}(g_0(\phi_0)) \rangle \]
A protocol

\[ A \xrightarrow{\text{sign}(r, skA)} B \]

\[ <\text{"ok"}, r> \]

Checks the signature
A quick introduction to the BC model

**A protocol**

\[ A \xrightarrow{\text{sign}(r,sk_A)} B \]

| Checks the signature |

\[ \langle \text{"ok"}, r \rangle \]

**In BC**

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A quick introduction to the BC model

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Names represent uniformly sampled bitstrings of length $\eta$

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& \quad < \text{"ok"}, \text{getmess}(g_0(\phi_0)) >
\end{align*}
\]
A quick introduction to the BC model

A protocol

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Names represent uniformly sampled bitstrings of length \( \eta \)

In BC

Protocols are modelled with sequences of terms:

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\begin{align*}
\phi_0 &:= \text{sign}(r, sk_A) \\
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\end{align*}
\]

Attacker inputs represented with non instantiated function symbol
How to reason on terms?
A first order logic built over a predicate:

\[ t_1 \sim t_2 \]
A quick introduction to the BC model

How to reason on terms?
A first order logic built over a predicate:

\[ t_1 \sim t_2 \]

For all \( \eta \), for all interpretations of free function symbols by PPT, any attacker can only distinguish between \( t_1 \) and \( t_2 \) with negligible probability.
How to make proofs
Logical rules allow to reason about \( \sim \):
How to make proofs
Logical rules allow to reason about $\sim$:

- for any term $t$, $t \sim t$
How to make proofs
Logical rules allow to reason about $\sim$:

- for any term $t$, $t \sim t$
- for any function symbol $f$ and terms $t_1, \ldots, t_n, t'_1, \ldots, t'_n$,

\[ t_1, \ldots, t_n \sim t'_1, \ldots, t'_n \Rightarrow f(t_1, \ldots, t_n) \sim f(t'_1, \ldots, t'_n) \]
A quick introduction to the BC model

How to make proofs
Logical rules allow to reason about ∼:

• for any term $t$, $t ∼ t$
• for any function symbol $f$ and terms $t_1, \ldots, t_n, t'_1, \ldots, t'_n$,

$$t_1, \ldots, t_n ∼ t'_1, \ldots, t'_n \Rightarrow f(t_1, \ldots, t_n) ∼ f(t'_1, \ldots, t'_n)$$

• transitivity, branching over conditionals, \ldots
EUF-CMA
For all terms $t$ such that $sk$ only appears in key position:

$$\text{checksign}(t, pk(sk)) \Rightarrow \bigvee_{\text{sign}(x, sk) \in St(t)} t = \text{sign}(x, sk)$$
A quick introduction to the BC model

**EUF-CMA**

For all terms $t$ such that $sk$ only appears in key position:

\[
\text{checksign}(t, pk(sk))) \Rightarrow \bigvee_{\text{sign}(x, sk) \in St(t)} t \doteq \text{sign}(x, sk) \sim true
\]
A quick introduction to the BC model

A reminder of our protocol

\[ \phi_0 := \text{sign}(r, sk_A) \]

\[ \phi_1 := \phi_0, \quad \text{if} \left( \text{checksign}(g_0(\phi_0), pk(sk_A)) \right) \text{then} \]

\[ < \text{“ok”}, \text{getmess}(g_0(\phi_0)) > \]
A quick introduction to the BC model

A reminder of our protocol

\[ \phi_0 := \text{sign}(r, sk_A) \]

\[ \phi_1 := \phi_0, \quad \text{if} \ ( \text{checksign}(g_0(\phi_0), pk(sk_A))) \text{ then} \]

\[ < "ok", \text{getmess}(g_0(\phi_0)) > \]

A security property

EUF-CMA \models \phi_1 \sim

\text{sign}(r, sk_A), \quad \text{if} \ ( \text{checksign}(g_0(\phi_0), pk(sk_A))) \text{ then} \]

\[ < "ok", r > \]
A compositional framework inside the computational model
A classical proof technique

A is trying to break protocol $\mathcal{P}$, while also having access to $\mathcal{Q}$.
A classical proof technique

$A$ is trying to break protocol $\mathcal{P}$, while also simulating $\mathcal{Q}$. 
A classical proof technique

$A$ is trying to break protocol $P$, while also \textbf{simulating} $Q$. 
A classical proof technique

$A$ is trying to break protocol $\mathcal{P}$, while also \textcolor{red}{simulating} $\mathcal{Q}$. 

$\mathcal{P}$

$A'$
A is trying to break protocol $\mathcal{P}$, while also simulating $\mathcal{Q}$. 
The main idea
If $A$ can simulate it, i.e. produce exactly all the same messages:

we remove $Q$ from the picture!
**The main idea**
If $A$ can simulate it, i.e. produce exactly all the same messages:

we remove $Q$ from the picture!

**The difficulty**
If $P$ and $Q$ share some secret key $sk$, $A$ cannot simulate messages which require $sk$. 
The main idea

Exemple for signatures

- \( P_{sk} \) may produce \( \text{sign}(< m, \text{"tag}_1\text{"} >, sk) \)
- \( Q_{sk} \) may produce \( \text{sign}(< m', \text{"tag}_2\text{"} >, sk) \)

The main idea

Exemple for signatures

- $P_{sk}$ may produce $\text{sign}(< m, \text{“tag} \_1 \text{”} >, sk)$
- $Q_{sk}$ may produce $\text{sign}(< m', \text{“tag} \_2 \text{”} >, sk)$

To prove $P$ while abstracting $Q$, the attacker must be able to produce $\text{sign}(< m', \text{“tag} \_2 \text{”} >, sk)$. 
The main idea

Exemple for signatures

- $P_{sk}$ may produce $\text{sign}(< m, "tag_1" >, sk)$
- $Q_{sk}$ may produce $\text{sign}(< m', "tag_2" >, sk)$

To prove $P$ while abstracting $Q$, the attacker must be able to produce $\text{sign}(< m', "tag_2" >, sk)$.

$\rightarrow$ We may give an oracle to the attacker, allowing to obtain $\text{sign}(< m', "tag_2" >, sk)$ but not $\text{sign}(< m, "tag_1" >, sk)$.
A is trying to break protocol $\mathcal{P}$, while simulating $\mathcal{Q}$ thanks to oracle $\mathcal{O}$.
A is trying to break protocol $\mathcal{P}$, while simulating $\mathcal{Q}$ thanks to oracle $\mathcal{O}$. 
The main idea

$A$ is trying to break protocol $P$, while simulating $Q$ thanks to oracle $O$. 

$P_{sk}$

$A' \ O$
The main idea

$A$ is trying to break protocol $P$, while simulating $Q$ thanks to oracle $O$. 

\[ P_{sk} \xrightarrow{} A^{O} \]
In the BC model

Classical Setting
To prove the security of $\mathcal{P}$ against $\mathcal{A}$, we define axioms $Ax$ that are sound for any PPT, and prove that $Ax \models \phi_{\mathcal{P}}$. 
In the BC model

**Classical Setting**
To prove the security of $P$ against $A$, we define axioms $Ax$ that are sound for any PPT, and prove that $Ax \models \phi_P$.

**New Axioms**
To prove $P$ against $AO$, we just have find axioms $Ax_O$ that are sounds for all PPTOM.
On an example
Signed DDH

\[ A(a, \text{sk}_A) \]

\[ \text{sign}(g^a, \text{sk}_A) \rightarrow \]

\[ x_B = g^a \]

\[ \text{sign}(\langle g^a, g^b \rangle, \text{sk}_B) \leftarrow \]

\[ x_A = g^b \]

\[ \text{sign}(\langle g^a, g^b \rangle, \text{sk}_A) \rightarrow \]

\[ k_A = x_A^a \]

\[ B(b, \text{sk}_B) \]

\[ k_B = x_B^b \]
A small DDH example

The security property:
\[ i \leq N (A(a_i, skA); \text{out}(k_A) || B(b_i, skB); \text{out}(k_B)) \]
\[ \sim \]
\[ i \leq N-1 (A(a_i, skA); \text{out}(k_A) || B(b_i, skB); \text{out}(k_B)) \]
\[ \| A(a_N, skA); \text{if } x_A = g^{b_N} \text{ then } \text{out}(k_{N,N}) \]
\[ \text{else if } x_A \notin \{g^{b_i}\}_{1 \leq i \leq N} \text{ then } \bot \]
\[ \| B(b_N, skB); \text{if } x_B = g^{a_N} \text{ then } \text{out}(k_{N,N}) \]
\[ \text{else if } x_B \notin \{g^{a_i}\}_{1 \leq i \leq N} \text{ then } \bot \]
A small DDH example

The final security property:
Let's assume the attacker can simulate

\[ i \leq N - 1 (A(a_i, skA); \textbf{out}(k_A)) \parallel B(b_i, skB); \textbf{out}(k_B)) \]

How to simulate the \(N - 1\) sessions?
A small DDH example

The final security property:
Let's assume the attacker can simulate

\[ \|^{i \leq N-1} (A(a_i, skA); \text{out}(k_A)) \| B(b_i, skB); \text{out}(k_B)) \]

We can simply prove:

\[
A(a_N, skA); \text{out}(k_A) \| B(b_N, skB); \text{out}(k_B)
\]

\[ \sim \]

\[
A(a_N, skA); \text{if } x_A = g^{b_N} \text{ then out}(k_{N,N})
\]

\[ \text{else if } x_A \notin \{g^{b_i}\}_{1 \leq i \leq N} \text{ then } \perp \]

\[
\| \ B(b_N, skB); \text{if } x_B = g^{a_N} \text{ then out}(k_{N,N})
\]

\[ \text{else if } x_B \notin \{g^{a_i}\}_{1 \leq i \leq N} \text{ then } \perp \]
A small DDH example

The final security property: Let's assume the attacker can simulate

\[ \|_{i \leq N-1} (A(a_i, skA); \text{out}(k_A) \| B(b_i, skB); \text{out}(k_B)) \]

We can simply prove:

\[ A(a_N, skA); \text{out}(k_A) \| B(b_N, skB); \text{out}(k_B) \sim \]

- \[ A(a_N, skA); \text{if } x_A = g^{b_N} \text{ then out}(k_{N,N}) \]
  - else if \( x_A \notin \{ g^{b_i} \}_{1 \leq i \leq N} \) then \( \perp \)

\[ \| B(b_N, skB); \text{if } x_B = g^{a_N} \text{ then out}(k_{N,N}) \]
  - else if \( x_B \notin \{ g^{a_i} \}_{1 \leq i \leq N} \) then \( \perp \)

\[ \leftrightarrow \text{ How to simulate the } N - 1 \text{ sessions?} \]
Simulating the sessions

What must the attacker be able to produce?
He must be able to start some $A$:

$$\forall 1 \leq i \leq N - 1. \ \text{sign}(g^{a_i}, skA)$$
Simulating the sessions

What must the attacker be able to produce?
He must be able to start some $A$:

$$\forall 1 \leq i \leq N - 1. \text{sign}(g^{a_i}, skA)$$

And for any DDH share $r$ he receives, he should be able to produce:

- $\forall 1 \leq i \leq N - 1. \text{sign}(<g^{a_i}, r>, skA)$
Simulating the sessions

**What must the attacker be able to produce?**

He must be able to start some $A$:

$$\forall 1 \leq i \leq N - 1. \ \text{sign}(g^{a_i}, skA)$$

And for any DDH share $r$ he receives, he should be able to produce:

- $$\forall 1 \leq i \leq N - 1. \ \text{sign}(< g^{a_i}, r >, skA)$$
- $$\forall 1 \leq i \leq N - 1. \ \text{sign}(< r, g^{b_i} >, skB)$$
Generic signing oracles

**T signing oracle**

\[ O_{T,sk}^{\text{sign}} : \text{input}(m) \]

\[ \text{if } T(m) \text{ then} \]

\[ \text{output}(\text{sign}(m, sk))) \]
Generic signing oracles

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\[ \mathcal{O}_{T,sk}^{\text{sign}} : \text{input}(m) \]

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Give the attacher access to \( \mathcal{O}_{T,skA}^{\text{sign}} \) and \( \mathcal{O}_{T,skB}^{\text{sign}} \) with:

\[ T(m) = \text{true} \iff \exists 1 \leq i \leq N - 1, r. \]

\[ \begin{align*}
  m &= g^{a_i} \\
  m &= < g^{a_i}, r > \\
  m &= < r, g^{b_i} >
\end{align*} \]
Generic signing oracles

**T signing oracle**

\[ O^{{\text{sign}}}_{T, sk} : \text{input}(m) \]

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Give the attacher access to \( O^{{\text{sign}}}_{T, skA} \) and \( O^{{\text{sign}}}_{T, skB} \) with:

\[ T(m) = \text{true} \iff \exists 1 \leq i \leq N - 1, r. \]

\[
\begin{cases} 
  m = g^{a_i} \\
  m = \langle g^{a_i}, r \rangle \\
  m = \langle r, g^{b_i} \rangle 
\end{cases}
\]

\( \leftrightarrow \) How to make the proof for such attackers?
**T-EUFCMA**

For any computable function $T$, for all terms $t$ such that $sk$ only appears in key position:

\[
\text{checksign}(t, pk(sk)) \Rightarrow T(\text{getmess}(t)) \land \bigvee_{\text{sign}(x, sk) \in \text{st}(t)} (t \doteq \text{sign}(x, sk)) \sim true
\]
The final proof

Assumption

\( \text{checksign}(t, pk(sk)) \implies \exists 1 \leq i \leq N - 1, r. \ \text{getmess}(t) \in \{g^{a_i}, < g^{a_i}, r >, < r, g^{b_i} >\} \)

\( \lor_{\text{sign}(x, sk) \in st(t)} (t \equiv \text{sign}(x, sk)) \)

\sim true
Assumption

\[
\text{checksign}(t, pk(sk))) \Rightarrow
\exists 1 \leq i \leq N - 1, r. \ \text{getmess}(t) \in \{g^{ai}, < g^{ai}, r >, < r, g^{bi} >\}
\vee_{\text{sign}(x, sk) \in \text{St}(t)}(t \doteq \text{sign}(x, sk)))
\sim true
\wedge \text{DDH} : g^{aN}, g^{bN}, g^{aNbN} \sim g^{aN}, g^{bN}, k_{N,N}
\]
The final proof

**Goal**

\[
A(a_N, skA); \text{out}(k_A) \| B(b_N, skB); \text{out}(k_B) \\
\sim \\
A(a_N, skA); \text{if } x_A = g^{b_N} \text{ then } \text{out}(k_{N,N}) \\
\text{else if } x_A \notin \{g^{b_i}\}_{1 \leq i \leq N} \text{ then } \bot \\
\| B(b_N, skB); \text{if } x_B = g^{a_N} \text{ then } \text{out}(k_{N,N}) \\
\text{else if } x_B \notin \{g^{a_i}\}_{1 \leq i \leq N} \text{ then } \bot
\]
The final proof

Synchronization

\[ A(a_N, skA); \text{ if } x_A = g^{b_N} \text{ then out}(g^{a_N b_N}) \]
\[ \text{ else if } x_A \notin \{g^{b_i}\}_{1 \leq i \leq N} \text{ then out}(x_A^{a_N}) \]
\[ \parallel B(b_N, skB); \text{ if } x_B = g^{a_N} \text{ then out}(g^{a_N b_N}) \]
\[ \text{ else if } x_B \notin \{g^{a_i}\}_{1 \leq i \leq N} \text{ then out}(x_B^{b_N}) \]
\sim

\[ A(a_N, skA); \text{ if } x_A = g^{b_N} \text{ then out}(k_{N,N}) \]
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$$\text{ else if } x_B \notin \{g^{a_i}\}_{1 \leq i \leq N} \text{ then out}(x_B^{b_N})$$
$$\sim$$

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The final proof

Synchronization

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A(a_N, skA); \text{ if } x_A = g^{b_N} \text{ then out}(g^{a_N b_N})
\]

\[
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\]

\[
\| B(b_N, skB); \text{ if } x_B = g^{a_N} \text{ then out}(g^{a_N b_N})
\]

\[
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\]

\[
\sim
\]

\[
A(a_N, skA); \text{ if } x_A = g^{b_N} \text{ then out}(k_{N,N})
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\[
\text{else if } x_B \notin \{g^{a_i}\}_{1 \leq i \leq N} \text{ then } \bot
\]
The final proof

**Synchronization**
Proof steps Split the conditionals into four cases and,

1. use DDH to show indistinguishability,
2. use T-EUF-CMA, to show that $x_A \notin \{g^i | 1 \leq i \leq N\}$ is never true (e.g., $\bot$ unreachable),
3. similar to (2);
4. similar to (2);
The final proof

**Synchronization**
Proof steps Split the conditionals into four cases and,

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Synchronization
Proof steps Split the conditionals into four cases and,

1. use DDH to show indistinguishability,

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Synchronization
Proof steps Split the conditionals into four cases and,

1. use DDH to show indistinguishability,
2. use T-EUF-CMA, to show that $x_A \notin \{g^{bi}\}_{1 \leq i \leq N}$ is never true (e.g, ⊥ unreachable),
3. similar to (2);
4. similar to (2);
Conclusion
The composition framework

- Composition results for parallel and sequential composition (in the BC model),
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- allows for long-term shared secrets and state-passing,
The composition framework

- Composition results for parallel and sequential composition (in the BC model),
- allows for long-term shared secrets and state-passing,
- allows for reduction from unbounded number of sessions to a single one,
Done and to do?

The composition framework

- Composition results for parallel and sequential composition (in the BC model),
- allows for long-term shared secrets and state-passing,
- allows for reduction from unbounded number of sessions to a single one,
- applied to key exchange (with key confirmations).
A tool
We are working on an interactive prover:

1. First allow to performe (un)-reachability proofs, (WIP)
A tool
We are working on an interactive prover:

1. First allow to performe (un)-reachability proofs, (WIP)
2. then integrate with indistinguishability proofs,
A tool
We are working on an interactive prover:

1. First allow to performe (un)-reachability proofs, (WIP)
2. then integrate with indistinguishability proofs,
3. and use the composition framework along with the tool to perform case studies.
Extra slides with too many details
Composition without replication
Let $C[_1, \ldots, _n]$ be a context such that the variable $k_i$ is bound in each hole $i$ and $P_1(x), \ldots, P_n(x)$ be parametrized protocols, such that all channels are disjoint. Given an oracle $\mathcal{O}$, with $\overline{n} \supset \mathcal{N}(C) \cap \mathcal{N}(P_1, \ldots, P_n)$, if, with $k'_1, \ldots, k'_n$ fresh names,

1. $C[\text{out}(1, k_1), \ldots, \text{out}(n, k_n)] \equiv C[\text{out}(1, k'_1), \ldots, \text{out}(n, k'_n)]$
2. $\nu \overline{n}. \text{in}(x). P_1(x) \parallel \ldots \parallel \text{in}(x). P_n(x)$ is $\mathcal{O}$-simulatable

Then $C[P_1(k_1), \ldots, P_n(k_n)] \equiv C[P_1(k'_1), \ldots, P_n(k'_n)]$
A core theorem

**Unbounded parallel Composition**
Let $O_r$ be an oracle and $Ax$ a set of axioms both parametrized by a sequence of names $\overline{s}$. Let $\overline{p}$ be a sequence of shared secrets, $P(\overline{x})$, $R(\overline{x}, \overline{y}, \overline{z})$ and $Q(\overline{x}, \overline{y})$ be parametrized protocols. If we have, for a sequence of names $\overline{lsid}$ and any integers $n$, if with $\overline{s} = \overline{lsid}_1, \ldots, \overline{lsid}_n$, $n$ copies of $\overline{lsid}$:

1. $\forall \ 1 \leq i \leq n, \nu \overline{p}.t_R(\overline{p}, \overline{lsid}_i, \overline{s})$ is $O_r$ simulatable.
2. $Ax$ is $O_r$ sound.
3. $Ax \models t_{P(\overline{p})} \sim t_{Q(\overline{p}, \overline{s})}$

Then, for any integer $n$:

\[
P(\overline{p}) \parallel !n R(\overline{p}, \overline{lsid}, \overline{s}) \approx Q(\overline{p}, \overline{s}) \parallel !n R(\overline{p}, \overline{lsid}, \overline{s})
\]
A core theorem

Unbounded parallel Composition
Let \( O_r \) be an oracle and \( Ax \) a set of axioms both parametrized by a sequence of names \( \bar{s} \). Let \( \bar{p} \) be a sequence of shared secrets, \( P(\bar{x}, \bar{y}) \) and \( Q(\bar{x}, \bar{y}, \bar{z}) \) be parametrized protocols. If we have, for sequences of names \( \overline{lsid_p}, \overline{lsid_q} \) and any integers \( n \), if with \( \bar{s} = \overline{lsid_p,1}, \ldots, \overline{lsid_p,n}, \ldots, \overline{lsid_q,n} \) sequences of copies of \( \overline{lsid_p}, \overline{lsid_q} \)

1. \( \forall 1 \leq i \leq n, \nu \bar{p}.t_P(\bar{p}, \overline{lsid_p,i}) \) is \( O_r \) simulatable.
2. \( \forall 1 \leq i \leq n, \nu \bar{p}.t_Q(\bar{p}, \overline{lsid_q,i}, \bar{s}) \) is \( O_r \) simulatable.
3. \( Ax \) is \( O_r \) sound.
4. \( Ax \models t_P(\bar{p}, \overline{lsid_p}) \sim t_Q(\bar{p}, \overline{lsid_q}, \bar{s}) \)

Then, for any integers \( n \):

\[ !_n P(\bar{p}, \overline{lsid_p}) \equiv_O !_n Q(\bar{p}, \bar{s}, \overline{lsid_q}) \]