Automates d'arbre

TD 4 : Logic and Alternation

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1 Logic

Exercise 1 : Decidability of the reducibility theory

Definition 1 *If $t \in \mathcal{T}(F, \mathcal{X})$ and $u \in \mathcal{T}(F)$, u encompasses t if there is a substitution σ such that $t\sigma$ is a subterm of u.*
The theory of reducibility associated with a set of term $S \subset \mathcal{T}(F, \mathcal{X})$ is the set of first-order formulas built on the unary predicates E_t, $t \in S$ and interpreted as the set of terms encompassing t.

1. Why is it called the reduciblity theory ?
2. Given a set of linear terms, show that its reducibility theory is decidable.

Exercise 2 : The power of Wsks
Produce formulae of WSkS for the following predicates :

- the set X has exactly two elements.
- the set X contains at least one string beginning with a 1.
- $x \leq_{lex} y$ where $\leq_{lex}$ is the lexicographic order on $\{1, ...k\}^*$.
- given a formula of WSkS ϕ with one free first-order variable, produce a formula of WSkS expressing that there is an infinity of words on $\{1, ..., k\}^*$ satisfying ϕ.

Exercise 3 : The limit of Wsks
Prove that the predicate $x = 1y$ is not definable in WSkS.

2 Alternation

Exercise 4 : SUCH AWA

Definition 2 *If $\mathcal{X}$ is a set of propositional variables, let $\mathbb{B}(\mathcal{X})$ be the set of positive propositional formulae on $\mathcal{X}$, i.e., formulae generated by the grammar $\phi ::= \bot \mid \top \mid \phi \vee \phi \mid \phi \wedge \phi$.*

Definition 3 *A AWA (Alternating Word Automata) is a tuple $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ where Σ is a finite set (alphabet), Q is a finite set (of states), $Q_0 \subset Q$ (initial states), $Q_f \subseteq Q$ (final states) and δ is a function from $Q \times \Sigma$ to $\mathbb{B}(Q)$ (transition function). A run of $\mathcal{A} = (Q, \Sigma, Q0, Q_f, \delta)$ on a word w is a tree t labelled by Q such that :*
— if $w = \varepsilon$, then $t = q_0$ with $q_0 \in Q_0$.
— if $w = a.w'$, then $t = q_0(t_1, \ldots, t_n)$ $q_0 \in Q_0$ and such that for all i, t_i is a run of w' on $(Q, \Sigma, q_i, Q_f, \delta)$ and $\{q_1, \ldots, q_n\} \models \delta(q_0, a)$.

Definition 4 *We say that a run is accepting if every leaf of the form q satisfies that $q \in Q_f$.*

1. Let $\Sigma = \{0, 1\}$ and $\mathcal{A} = (Q, \Sigma, q_0, Q_f, \delta)$ the AWA such that $Q = \{q_0, q_1, q_2, q_3, q_4, q'_1, q'_2\}$, $Q_f = \{q_0, q_1, q_2, q_3, q_4\}$ and :

$$\begin{array}{lll} \delta = \{ & q_0 0 \longrightarrow (q_0 \wedge q_1) \vee q'_1 & q_0 1 \longrightarrow q_0 \\ & q_1 0 \longrightarrow q_2 & q_1 1 \longrightarrow \top \\ & q_2 0 \longrightarrow q_3 & q_2 1 \longrightarrow q_3 \\ & q_3 0 \longrightarrow q_4 & q_3 1 \longrightarrow q_4 \\ & q_4 0 \longrightarrow \top & q_4 1 \longrightarrow \top \\ & q'_1 0 \longrightarrow q'_1 & q'_1 1 \longrightarrow q'_2 \\ & q'_2 0 \longrightarrow q'_2 & q'_2 1 \longrightarrow q'_1 \} \end{array}$$

   Give an example of an accepting computation of $\mathcal{A}$ on $w = 00101$ and an example of a non accepting computation of $\mathcal{A}$ on w.

2. Prove that for all AWA, we can compute in exponential time a non-deterministic automaton which accepts the same language.

3. Show how to reduce the emptiness problem for an AWA on a one letter alphabet $\{a\}$ with formalas that are in positive disjunctive normal form to the emptiness problem of a tree automaton .

4. Show how to reduce the emptiness problem for a tree automaton to the emptiness problem of an AWA on a one letter alphabet $\{a\}$. Conclude on the complexity of the emptiness problem for an AWA on a one letter alphabet.