# Automates d'arbre

## TD n°1 : Recognizable Tree Languages and Finite Tree Automata

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**Exercise 1 : First constructions of Tree Automatas**

Let $\mathcal{F} = \{f(2), g(1), a(0)\}$. Give a DFTA and a top-down DFTA for the set $G(t)$ of ground instances of the term $t = f\big(f(a,x), g(y)\big)$ which is defined by :

$$G(t) = \Big\{ f\big(f(a,u), g(v)\big) \mid u, v \in T(\mathcal{F}) \Big\}$$

**Solution:**

- top-down DFTA : $Q = \{q_{f,1}, q_{f,2}, q_g, q_a, q_\top\}$, $I = \{q_{f,1}\}$ and $\Delta =$
  - ⋆ $q_{f,1}(f(x,y)) \longrightarrow f(q_{f,2}(x), q_g(y))$
  - ⋆ $q_{f_2}(f(x,y)) \longrightarrow f(q_a(x), q_\top(y))$
  - ⋆ $q_g(g(x)) \longrightarrow g(q_\top(x))$
  - ⋆ $q_a(a) \longrightarrow a$
  - ⋆ $q_\top(f(x,y)) \longrightarrow f(q_\top(x), q_\top(y))$
  - ⋆ $q_\top(g(x)) \longrightarrow g(q_\top(x))$
  - ⋆ $q_\top(a) \longrightarrow a$
- DFTA : $Q = \{q_a, q_f, q_g, q_\top, q_\bot\}$, $F = \{q_\top\}$ and $\Delta =$
  - ⋆ $a \longrightarrow q_a$
  - ⋆ $f(q_a, q) \longrightarrow q_f$ for all $q \in Q$
  - ⋆ $g(q) \longrightarrow q_g$ for all $q \in Q$
  - ⋆ $f(q_f, q_g) \longrightarrow q_\top$
  - ⋆ $f(q, q') \longrightarrow q_\bot$ for all $(q, q') \neq (q_a, \_), (q_f, q_g)$

**Exercise 2 : What is recognizable by an FTA ?**

Are the following tree languages recognizable (by a bottom-up FTA) ?

- $\mathcal{F} = \{g(1), a(0)\}$ and $L$ the set of ground terms of even height.
- $\mathcal{F} = \{f(2), g(1), a(0)\}$ and $L$ the set of ground terms of even height.

**Solution:**

- Yes.
- No. Remark that the pumping lemma does not apply ! Assume that it is recognizable by a NFTA with $n$ states. Define :

$$t_n = f(g^{2n+1}(a), g^{2n+2}(a))$$

It has height $2n+2$ and so belongs to this language. So there exists an accepting run $\rho$ for $t_n$. By the pigeonhole principle, there exists $k < k'$ such that $r(1.1^k) = r(1.1^{k'})$ and from that we deduce that for all $p \in \mathbb{N}$, the tree

$$t_{n,p} = f(g^{2n+1+p(k'-k)}(a), g^{2n+2}(a))$$

also has an accepting run. But $t_{n,2}$ has height $2(n + k' - k) + 1$ which is odd. Contradiction.

**Exercise 3 : Bottom-up vs Top-down**

1) Recall why bottom-up NFTAs, bottom-up DTAs and top-down NFTAs have the same expressiveness.
2) Let $\mathcal{F} = \{f(2), g(1), a(0)\}$. Give a DFTA and a top-down NFTA for the set $M(t)$ of terms which have a ground instance of the term $t = f(a, g(x))$ as a subterm, ie. $M(t) = \{C[f(a, g(u))] \mid C \in \mathcal{C}(\mathcal{F}), u \in T(\mathcal{F})\}$.
3) Show that NFTAs and top-down DFTAs do not have the same expressiveness.

**Solution:**

- top-down NFTA : $Q = \{q_0, q_\perp, q_a, q_g\}$, $I = \{q_0\}$ and $\Delta =$
  - ⋆ $q_0(f(x, y)) \longrightarrow f(q_\perp(x), q_0(y))$
  - ⋆ $q_0(f(x, y)) \longrightarrow f(q_0(x), q_\perp(y))$
  - ⋆ $q_\perp(f(x, y)) \longrightarrow f(q_\perp(x), q_\perp(y))$
  - ⋆ $q_\perp(g(x)) \longrightarrow g(q_\perp(x))$
  - ⋆ $q_\perp(a) \longrightarrow a$
  - ⋆ $q_0(g(x)) \longrightarrow g(q_0(x))$
  - ⋆ $q_0(f(x, y)) \longrightarrow f(q_a(x), q_g(y))$
  - ⋆ $q_a(a) \longrightarrow a$
  - ⋆ $q_g(g(x)) \longrightarrow g(q_\perp(x))$
- DFTA : $Q = \{q_a, q_g, q_\top, q_\perp\}$, $F = \{q_\top\}$ and $\Delta =$
  - ⋆ $a \longrightarrow q_a$
  - ⋆ $g(q_\top) \longrightarrow q_\top$
  - ⋆ $g(q) \longrightarrow q_g$ with $q \neq q_\top$
  - ⋆ $f(q, q') \longrightarrow q_\top$ if $(q, q') = (q_a, q_g)$ or $q = q_\top$ or $q' = q_\top$
  - ⋆ $f(q, q') \longrightarrow q_\perp$ else
- Let's assume M(t) can be recognized by a top-down DFTA $\mathcal{A}$. We consider two terms $t_1 = f(t, a)$ and $t_2 = f(a, t)$. $\mathcal{A}$ must start with the same transition on both terms, let's say $q_0(f(x, y)) \longrightarrow f(q_L(x), q_R(y))$. Then, there is an accepting run for $q_R(a)$ because $t_1$ in M(t), and conversely for $q_L(a)$. Finally, $\mathcal{A}$ accepts $f(a, a) \notin M(t)$.

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**Bonus exercice : Satisfiability**

Let $\mathcal{F} = \{and(2), or(2), not(1), 0(0), 1(0), x(0)\}$. A ground term over $\mathcal{F}$ can then be viewed as a boolean formula over $x$.

1) Give an NFTA which recognizes the set of satisfiable boolean formulae over $x$.

Let $\mathcal{F} = \{and(2), or(2), not(1), 0(0), 1(0), x_1(0), \ldots, x_n(0)\}$, i.e we now handle $n$ variables instead of a single one. The same variable may appear several times in a formula, and should be evaluated consistently.

2) Give an NFTA which recognizes the set of satisfiable boolean formulae over $x_1, \ldots, x_n$.

**Solution:**