Automates d'arbre

TD n°6 : PDL

Charlie Jacomme

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Exercise 1 : Warm up PDL

Definition 1 (PDL)
The syntax is the following :

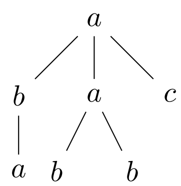
$$\phi ::= a \mid \top \mid \neg\phi \mid \phi \vee \phi \mid \langle\pi\rangle\,\phi \qquad \textit{(position formulae)}$$
$$\pi ::= \downarrow \mid \rightarrow \mid \pi^{-1} \mid \pi;\pi \mid \pi + \pi \mid \pi^* \mid \phi? \qquad \textit{(path formulae)}$$

The semantic is defined this way : let t be a tree, we define $[\![\phi]\!]_t$ (resp. $[\![\pi]\!]_t$) as a set of positions of t (resp. a relation on positions of t) by induction on the size of ϕ (resp. π) :

$$\begin{aligned}
[\![a]\!]_t &= \{w \in Pos(t) \mid t(w) = a\} & [\![\downarrow]\!]_t &= \{(w, w.i) \mid w, w.i \in Pos(t)\} \\
[\![\top]\!]_t &= Pos(t) & [\![\rightarrow]\!]_t &= \{(w.i, w.(i+1)) \mid w.i, w.(i+1) \in Pos(t)\} \\
[\![\neg\phi]\!]_t &= Pos(t) \setminus [\![\phi]\!]_t & [\![\pi^{-1}]\!]_t &= [\![\pi]\!]_t^{-1} \\
[\![\phi_1 \vee \phi_2]\!]_t &= [\![\phi_1]\!]_t \cup [\![\phi_2]\!]_t & [\![\pi_1;\pi_2]\!]_t &= [\![\pi_2]\!]_t \circ [\![\pi_1]\!]_t \\
[\![\langle\pi\rangle\,\phi]\!]_t &= [\![\pi]\!]_t^{-1}([\![\phi]\!]_t) & [\![\pi_1 + \pi_2]\!]_t &= [\![\pi_1]\!]_t \cup [\![\pi_2]\!]_t \\
[\![\pi^*]\!]_t &= [\![\pi]\!]_t^* & [\![\phi?]\!]_t &= \Delta_{[\![\phi]\!]_t} = \{(w, w) \mid w \in [\![\phi]\!]_t\}
\end{aligned}$$

Let t be a tree and w, $w' \in Pos(t)$. We note :
— $t, w \models \phi$ *if* $w \in [\![\phi]\!]_t$
— $t \models \phi$ *if* $t, \epsilon \models \phi$ *and we say that* t *satisfies* ϕ
— $t, w, w' \models \pi$ *if* $(w, w') \in [\![\pi]\!]_t$

Let t be the tree :



Which formulae are satisfied by t ?

1. $\phi_1 = \neg a \vee \langle\downarrow\rangle \Big(\neg \langle\leftarrow\rangle \top \wedge b \wedge \langle\rightarrow^*\rangle\,(c \wedge \neg \langle\rightarrow\rangle \top)\Big)$
2. $\phi_2 = \neg a \vee \langle\downarrow\rangle \Big(\neg \langle\leftarrow\rangle \top \wedge b \wedge \langle(\rightarrow; c?)^*\rangle\,(\neg \langle\rightarrow\rangle \top)\Big)$
3. $\phi_3 = \langle(a?;\downarrow)^*\rangle\,(a \wedge \neg \langle\downarrow\rangle \top)$

Solution:

1. yes
2. no
3. no

Exercise 2 : The power of PDL ?

Give a translation of PDL in MSO which preserves models. That is, given a position formula ϕ (resp. a path formula π), construct a MSO formula $\tilde{\phi}$ (resp. $\tilde{\pi}$) whose set of free variable is $\{X_a \mid a \in \mathcal{F}\} \cup \{x\}$ (resp. $\{X_a \mid a \in \mathcal{F}\} \cup \{x, y\}$) such that $t, w \models \phi$ iff $(P_a(t))_{a\in\mathcal{F}}, w \models \tilde{\phi}$ (resp. $t, w, w' \models \pi$ iff $(P_a(t))_{a\in\mathcal{F}}, w, w' \models \tilde{\pi}$) where $P_a(t) = \{w \in Pos(t) \mid t(w) = a\}$.

Solution:

By induction on the size of the formula :

- $\phi = b \in \mathcal{F} : \tilde{\phi}((X_a)_{a\in\mathcal{F}}, x) \doteq x \in X_b$
- $\phi = \phi_1 \wedge \phi_2 : \tilde{\phi}((X_a)_{a\in\mathcal{F}}, x) \doteq \tilde{\phi_1}((X_a)_{a\in\mathcal{F}}, x) \wedge \tilde{\phi_2}((X_a)_{a\in\mathcal{F}}, x)$
- idem for $\vee$ $\neg$ and $\top$
- $\phi = \langle\pi\rangle\phi' : \tilde{\phi}((X_a)_{a\in\mathcal{F}}, x) \doteq \exists y.\, \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \wedge \tilde{\phi}'((X_a)_{a\in\mathcal{F}}, y)$
- $\pi = \downarrow : \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq child(x, y)$
- $\pi = \rightarrow : \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq next_sibling(x, y)$
- $\pi = \pi'^{-1} : \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq \tilde{\pi}'((X_a)_{a\in\mathcal{F}}, y, x)$
- $\pi = \pi_1; \pi_2 : \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq \exists z.\, \tilde{\pi}_1((X_a)_{a\in\mathcal{F}}, x, z) \wedge \tilde{\pi}_2((X_a)_{a\in\mathcal{F}}, z, y)$
- $\pi = \pi_1 + \pi_2 : \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq \tilde{\pi}_1((X_a)_{a\in\mathcal{F}}, x, y) \vee \tilde{\pi}_2((X_a)_{a\in\mathcal{F}}, x, y)$
- $\pi = \pi'^{*} : \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq \forall X\, (x \in X \wedge \forall z_1, z_2.\, ((z_1 \in X \wedge \tilde{\pi}'((X_a)_{a\in\mathcal{F}}, z_1, z_2)) \Rightarrow z_2 \in X)) \Rightarrow y \in X$
- $\pi = ?\phi : \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq (x = y) \wedge \tilde{\phi}((X_a)_{a\in\mathcal{F}}, x)$

Exercise 3 : The exerice we won't have time for

Fix an alphabet $\mathcal{F}$. Give a PDL formula π such that :

- for all tree t and all position p of t, there exists exactly one position q of t such that $(p, q) \in [\![\pi]\!]_t$ (π defines a function on positions).
- for all tree t and position p of t, $(p, q) \in [\![\pi^*]\!]_t$ iff q is a position of t such that $t(q) = t(p)$.

Solution:

Hard. There is no point to give you an answer. It will take you more time to understand it than find one by yourself.