Exercise 1: Extensions

Definition 1 (extension encoding)
Let $t$ be an unranked tree on $\Sigma$. Let $F_{ext}^{\Sigma} = \{ @(2) \} \cup \{ a(0) \mid a \in \Sigma \}$. We define the ranked tree $ext(t)$ by induction on the size of $t$ by:

- for $a \in \Sigma$, $ext(a) = a$
- if $t = a(t_1, ..., t_n)$ with $n \geq 1$, $ext(t) = @(ext(a(t_1, ..., t_{n-1})), ext(t_n))$

that is $ext(a(t_1, ..., t_n))$ is equal to:

\[
\begin{array}{c}
\hat{a} \\
@ \\
\hat{a} \\
\hat{a} \\
\end{array}
\]

Give the extension encoding of:

\[
\begin{array}{c}
\hat{a} \\
\hat{a} \\
\hat{a} \\
\end{array}
\]

Exercise 2: The soundess of the extension
Let $L$ be a language of unranked trees. Prove that $L$ is recognizable by a NFHA iff $ext(L)$ is recognizable by a NFTA.

Exercise 3: Complexity
Show that the emptiness problem for NFHA(NFA) is in PTIME.
**Exercise 4:** SUCH AWA

**Definition 2** If $X$ is a set of propositional variables, let $\mathbb{B}(X)$ be the set of positive propositional formulae on $X$, i.e., formulae generated by the grammar $\phi ::= \bot \mid T \mid x \in X \mid \phi \vee \phi \mid \phi \wedge \phi$.

**Definition 3** A AWA (Alternating Word Automaton) is a tuple $A = (Q, \Sigma, Q_0, Q_f, \delta)$ where $\Sigma$ is a finite set (alphabet), $Q$ is a finite set (of states), $Q_0 \subseteq Q$ (initial states), $Q_f \subseteq Q$ (final states) and $\delta$ is a function from $Q \times \Sigma$ to $\mathbb{B}(Q)$ (transition function). A run of $A = (Q, \Sigma, Q_0, Q_f, \delta)$ on a word $w$ is a tree $t$ labelled by $Q \times \mathbb{N}$ such that:
- if $w = \varepsilon$, then $t = (q_0, 0)$ with $q_0 \in Q_0$.
- if $w = a.w'$, then $t = (q_0, k)(t_1, \ldots, t_n)$ where $k$ is the length of $w$, $q_0 \in Q_0$ and such that for all $i$, $t_i$ is a run of $w'$ on $(Q, \Sigma, \{q_i\}, Q_f, \delta)$ for some $q_i$ satisfying $\{q_1, \ldots, q_n\} \models \delta(q_0, a)$.

**Definition 4** We say that a run is accepting if every leaf of the form $(q, 0)$ satisfies that $q \in Q_f$. Notice that a run may have leaves of the form $(q, i)$ with $i > 0$ if $\emptyset \models \delta(q_0, a)$. Those leaves are considered as ‘success’ leaves in this semantic. The language of a AWA is the set of words which have an accepting run.

1. Show how to reduce the emptiness problem for an AWA on a one letter alphabet $\{a\}$ with formalas that are in positive disjunctive normal form to the emptiness problem of a tree automaton.
2. Show how to reduce the emptiness problem for a tree automaton to the emptiness problem of an AWA on a one letter alphabet $\{a\}$. Conclude on the complexity of the emptiness problem for an AWA on a one letter alphabet.

**Homework for next week : Membership**

1. Recall the complexity of the uniform membership problem for DFTAs, NFTAs and NF-HAs.
2. Prove that (HarderUMembership):
   - **Instance**: an NFHA $A$ where the horizontal languages are given by AWA (and not finite automata) and a word $w$
   - **Question**: $w \in L(A)$?
   is in NP.