Automates d'arbre

TD n°6 : Alternation and PDL

Charlie Jacomme

October 17, 2017

Exercise 1 : SUCH AWA

Definition 1 *If $\mathcal{X}$ is a set of propositional variables, let $\mathbb{B}(\mathcal{X})$ be the set of positive propositional formulae on $\mathcal{X}$, i.e., formulae generated by the grammar $\phi ::= \bot \mid \top \mid \phi \vee \phi \mid \phi \wedge \phi$.*

Definition 2 *A AWA (Alternating Word Automata) is a tuple $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ where Σ is a finite set (alphabet), Q is a finite set (of states), $Q_0 \subseteq Q$ (initial states), $Q_f \subseteq Q$ (final states) and δ is a function from $Q \times \Sigma$ to $\mathbb{B}(Q)$ (transition function). A run of $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ on a word w is a tree t labelled by $Q \times \mathbb{N}$ such that :*
— *if $w = \varepsilon$, then $t = (q_0, 0)$ with $q_0 \in Q_0$.*
— *if $w = a.w'$, then $t = (q_0, k)(t_1, \ldots, t_n)$ where k is the length of w, $q_0 \in Q_0$ and such that for all i, t_i is a run of w' on $(Q, \Sigma, \{q_i\}, Q_f, \delta)$ for some q_i satisfying $\{q_1, \ldots, q_n\} \models \delta(q_0, a)$.*

Definition 3 *We say that a run is accepting if every leaf of the form $(q, 0)$ satisfies that $q \in Q_f$. Notice that a run may have leaves of the form (q, i) with $i > 0$ if $\varnothing \models \delta(q_0, a)$. Those leaves are considered as 'success' leaves in this semantic. The language of a AWA is the set of words which have an accepting run.*

1. Show how to reduce the emptiness problem for an AWA on a one letter alphabet $\{a\}$ whith formalas that are in positive disjunctive normal form to the emptiness problem of a tree automaton .
2. Show how to reduce the emptiness problem for a tree automaton to the emptiness problem of an AWA on a one letter alphabet $\{a\}$. Conclude on the complexity of the emptiness problem for an AWA on a one letter alphabet.

Solution:

1. Given $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ an AWA we construct an NFTA of the form $(Q, \{f_k(k) \mid 0 \leq k \leq n\}, F, \Delta')$ with $F = Q_0$:
$$\delta(q, a) = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{k_i} (q_{i,j}, i) \Rightarrow \forall i, f_i(q_{i,1}, ..., q_{i,k_i}) \longrightarrow q \in \delta'$$
2. Given $\mathcal{A} = (Q, \mathcal{F}, Q_f, \delta)$ an NFTA, we construct the AWA $\mathcal{A}' = (Q \times \mathcal{F}, \{a\}, I, F, \delta')$ with :
$$F = \{(q, f) \mid f \longrightarrow q \in \Delta\}$$
$$I = \{(q, f) \mid q \in Q_f\}$$
$$\delta((q, f), a) = \bigvee_{f(q_1,\ldots,q_n) \longrightarrow q \in \delta} \bigwedge_{i=1}^{n} \bigvee_{f_j \in \mathcal{F})} ((q_i, f_j), i)$$
.
We deduce that emptiness for AWA on singleton alphabet is P-hard.

Exercise 2 : Membership

1. Recall the complexity of the uniform membership problem for DFTAs, NFTAs and NFHAs.
2. Prove that **(AlternatingUMembership)** :
   Instance : an AWA $\mathcal{A}$ and a word w
   Question : $w \in L(\mathcal{A})$?
   is in PTIME.

3. Prove that **(HarderUMembership)** :
   Instance : an NFHA $\mathcal{A}$ where the horizontal languages are given by AWA (and not finite automata) and a word w
   Question : $w \in L(\mathcal{A})$?
   is in NP.
4. Let Φ be a propositional formula in CNF with n variables $x_1, ..., x_n$. Construct, in polynomial time, an AWA $\mathcal{A}_\Phi$ whose language is $\{w \in \{0,1\}^n \mid w \models \Phi \text{ i.e. } \Phi_{[x_i \leftarrow w_i]} = \top\}$.
5. Deduce that membership for NFHA where horizontal languages are given by AWA is NP-complete.

Solution:

1. In the case of DFTA, from a term t and the automaton A, we can compute a run in $O(|t| + |A|)$. In the nondeterministic case, the idea is similar to the word case : the algorithm determinizes along the computation, i.e. for each node of the term, we compute the set of reached states. The complexity of this algorithm will be in $O(|t| \times |A|)$.
   For NFHAs, the idea is similar to the NFTA, and we obtain a PTIME algorithm.
2. Let n be the size of w. Define S_i with $0 \leq i \leq n$ by decreasing induction :
   — $S_n = Q_f$
   — $S_i = \{q \in Q \mid \exists q, w_i \to \phi \in \mathcal{A}, S_{i+1} \models \phi\}$
   Then $w \in L(A)$ iff $I \cap S_0 \neq \varnothing$. All this can be done in PTIME.
3. First, guess a run i.e. a coloring ρ by states of your tree. Second, check this is an accepting run i.e. for all position p of t, check that there exists a transition of the form $t(p)(L) \longrightarrow \rho(p)$, that $\rho(p.1)...\rho(p.k) \in L$ (which can be done in P by exercise 1) and $\rho(\epsilon)$ is a final state.
4. Let $\Phi = \bigwedge_{j=1}^{m} C_j$ with C_j clauses. The expected AWA is the following :
   - $Q = \{q_0\} \cup \{q_j^{\alpha,k} \mid 1 \leq j \leq m, 1 \leq k \leq n, \alpha \in \{0,1\}\}$
   - $I = \{q_0\}$
   - $F = \{q_j^{1,n} \mid 1 \leq j \leq m\}$
   - $\Delta =$
     * $q_0, 1 \longrightarrow \bigwedge_{j \mid x_1 \in C_j} q_j^{1,1} \wedge \bigwedge_{j \mid x_1 \notin C_j} q_j^{0,1}$
     * $q_0, 0 \longrightarrow \bigwedge_{j \mid \neg x_1 \in C_j} q_j^{1,1} \wedge \bigwedge_{j \mid \neg x_1 \notin C_j} q_j^{0,1}$
     * $q_j^{1,k}, \_ \longrightarrow q_1^{1,k+1}$ for all $k < n$ and all j
     * $q_1^{0,k}, 1 \longrightarrow q_j^{1,k+1}$ for all j and all $k < n$ such that $x_k \in C_j$
     * $q_1^{0,k}, 1 \longrightarrow q_j^{0,k+1}$ for all j and all $k < n$ such that $x_k \notin C_j$
     * $q_1^{0,k}, 0 \longrightarrow q_j^{1,k+1}$ for all j and all $k < n$ such that $\neg x_k \in C_j$
     * $q_1^{0,k}, 0 \longrightarrow q_j^{0,k+1}$ for all j and all $k < n$ such that $\neg x_k \notin C_j$
5. NP-hard : we reduce SAT. Let Φ in CNF. We construct $\tilde{\mathcal{A}}_\Phi$ this way : $Q = \{0, 1, q_f\}$, $F = \{q_f\}$, $\Sigma = \{a, \#\}$ and $\Delta = \{\#(\epsilon) \longrightarrow 1, \#(\epsilon) \longrightarrow 0, a(\mathcal{A}_\Phi) \longrightarrow q_f\}$ where $\mathcal{A}_\Phi$ is from the previous question. Then Φ is satisfiable iff $a(\#, ..., \#) \in L(\tilde{\mathcal{A}}_\Phi)$.

Definition 4 (PDL)
The syntax is the following :

$$\phi ::= a \mid \top \mid \neg\phi \mid \phi \vee \phi \mid \langle\pi\rangle\,\phi \qquad \textit{(position formulae)}$$
$$\pi ::= \downarrow \mid \rightarrow \mid \pi^{-1} \mid \pi;\pi \mid \pi+\pi \mid \pi^{*} \mid \phi? \qquad \textit{(path formulae)}$$

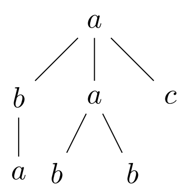
The semantic is defined this way : let t be a tree, we define $[\![\phi]\!]_t$ (resp. $[\![\pi]\!]_t$) as a set of positions of t (resp. a relation on positions of t) by induction on the size of ϕ (resp. π) :

$$\begin{aligned}
[\![a]\!]_t &= \{w \in Pos(t) \mid t(w) = a\} & [\![\downarrow]\!]_t &= \{(w, w.i) \mid w, w.i \in Pos(t)\}\\
[\![\top]\!]_t &= Pos(t) & [\![\rightarrow]\!]_t &= \{(w.i, w.(i+1)) \mid w.i, w.(i+1) \in Pos(t)\}\\
[\![\neg\phi]\!]_t &= Pos(t) \setminus [\![\phi]\!]_t & [\![\pi^{-1}]\!]_t &= [\![\pi]\!]_t^{-1}\\
[\![\phi_1 \vee \phi_2]\!]_t &= [\![\phi_1]\!]_t \cup [\![\phi_2]\!]_t & [\![\pi_1;\pi_2]\!]_t &= [\![\pi_2]\!]_t \circ [\![\pi_1]\!]_t\\
[\![\langle\pi\rangle\,\phi]\!]_t &= [\![\pi]\!]_t^{-1}([\![\phi]\!]_t) & [\![\pi_1+\pi_2]\!]_t &= [\![\pi_1]\!]_t \cup [\![\pi_2]\!]_t\\
[\![\pi^{*}]\!]_t &= [\![\pi]\!]_t^{*} & [\![\phi?]\!]_t &= \Delta_{[\![\phi]\!]_t} = \{(w,w) \mid w \in [\![\phi]\!]_t\}
\end{aligned}$$

Let t be a tree and w, $w' \in Pos(t)$. We note :
— $t, w \models \phi$ *if* $w \in [\![\phi]\!]_t$
— $t \models \phi$ *if* $t, \epsilon \models \phi$ *and we say that t satisfies ϕ*
— $t, w, w' \models \pi$ *if* $(w, w') \in [\![\pi]\!]_t$

Exercise 3 : Warm up

Let t be the tree :



Which formulae are satisfied by t ?

1. $\phi_1 = \neg a \vee \langle\downarrow\rangle \Big(\neg\langle\leftarrow\rangle\top \wedge b \wedge \langle\rightarrow^{*}\rangle\,(c \wedge \neg\langle\rightarrow\rangle\top)\Big)$
2. $\phi_2 = \neg a \vee \langle\downarrow\rangle \Big(\neg\langle\leftarrow\rangle\top \wedge b \wedge \langle(\rightarrow; c?)^{*}\rangle\,(\neg\langle\rightarrow\rangle\top)\Big)$
3. $\phi_3 = \langle(a?;\downarrow)^{*}\rangle\,(a \wedge \neg\langle\downarrow\rangle\top)$

Solution:

1. yes
2. no
3. no

Exercise 4 : The power of PDL ?

Give a translation of PDL in MSO which preserves models. That is, given a position formula ϕ (resp. a path formula π), construct a MSO formula $\tilde{\phi}$ (resp. $\tilde{\pi}$) whose set of free variable is $\{X_a \mid a \in \mathcal{F}\} \cup \{x\}$ (resp. $\{X_a \mid a \in \mathcal{F}\} \cup \{x, y\}$) such that $t, w \models \phi$ iff $(P_a(t))_{a\in\mathcal{F}}, w \models \tilde{\phi}$ (resp. $t, w, w' \models \pi$ iff $(P_a(t))_{a\in\mathcal{F}}, w, w' \models \tilde{\pi}$) where $P_a(t) = \{w \in Pos(t) \mid t(w) = a\}$.

Solution:

By induction on the size of the formula :

- $\phi = b \in \mathcal{F}$: $\tilde{\phi}((X_a)_{a\in\mathcal{F}}, x) \doteq x \in X_b$
- $\phi = \phi_1 \wedge \phi_2$: $\tilde{\phi}((X_a)_{a\in\mathcal{F}}, x) \doteq \tilde{\phi_1}((X_a)_{a\in\mathcal{F}}, x) \wedge \tilde{\phi_2}((X_a)_{a\in\mathcal{F}}, x)$

- idem for $\vee$ $\neg$ and $\top$
- $\phi = \langle \pi \rangle \phi'$: $\tilde{\phi}((X_a)_{a\in\mathcal{F}}, x) \doteq \exists y.\, \tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \wedge \tilde{\phi}'((X_a)_{a\in\mathcal{F}}, y)$
- $\pi = \downarrow$: $\tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq child(x, y)$
- $\pi = \rightarrow$: $\tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq next_s ibling(x, y)$
- $\pi = \pi'^{-1}$: $\tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq \tilde{\pi}'((X_a)_{a\in\mathcal{F}}, y, x)$
- $\pi = \pi_1; \pi_2$: $\tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq \exists z.\, \tilde{\pi}_1((X_a)_{a\in\mathcal{F}}, x, z) \wedge \tilde{\pi}_2((X_a)_{a\in\mathcal{F}}, z, y)$
- $\pi = \pi_1 + \pi_2$: $\tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq \tilde{\pi}_1((X_a)_{a\in\mathcal{F}}, x, y) \vee \tilde{\pi}_2((X_a)_{a\in\mathcal{F}}, x, y)$
- $\pi = \pi'^*$: $\tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq \forall X\, (x \in X \wedge \forall z_1, z_2.\, ((z_1 \in X \wedge \tilde{\pi}'((X_a)_{a\in\mathcal{F}}, z_1, z_2)) \Rightarrow z_2 \in X)) \Rightarrow y \in X$
- $\pi = ?\phi$: $\tilde{\pi}((X_a)_{a\in\mathcal{F}}, x, y) \doteq (x = y) \wedge \tilde{\phi}((X_a)_{a\in\mathcal{F}}, x)$

Exercise 5 : The exerice we won't have time for

Fix an alphabet $\mathcal{F}$. Give a PDL formula π such that :

- for all tree t and all position p of t, there exists exactly one position q of t such that $(p, q) \in [\![\pi]\!]_t$ (π defines a function on positions).
- for all tree t and position p of t, $(p, q) \in [\![\pi^*]\!]_t$ iff q is a position of t such that $t(q) = t(p)$.

Solution:

Hard. There is no point to give you an answer. It will take you more time to understand it than find one by yourself.