Automates d'arbre

TD n°6 : Alternation and PDL

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Exercise 1 : SUCH AWA

Definition 1 *If $\mathcal{X}$ is a set of propositional variables, let $\mathbb{B}(\mathcal{X})$ be the set of positive propositional formulae on $\mathcal{X}$, i.e., formulae generated by the grammar $\phi ::= \perp \mid \top \mid \phi \vee \phi \mid \phi \wedge \phi$.*

Definition 2 *A AWA (Alternating Word Automata) is a tuple $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ where Σ is a finite set (alphabet), Q is a finite set (of states), $Q_0 \subseteq Q$ (initial states), $Q_f \subseteq Q$ (final states) and δ is a function from $Q \times \Sigma$ to $\mathbb{B}(Q)$ (transition function). A run of $\mathcal{A} = (Q, \Sigma, Q_0, Q_f, \delta)$ on a word w is a tree t labelled by $Q \times \mathbb{N}$ such that :*
- *if $w = \varepsilon$, then $t = (q_0, 0)$ with $q_0 \in Q_0$.*
- *if $w = a.w'$, then $t = (q_0, k)(t_1, \ldots, t_n)$ where k is the length of w, $q_0 \in Q_0$ and such that for all i, t_i is a run of w' on $(Q, \Sigma, \{q_i\}, Q_f, \delta)$ for some q_i satisfying $\{q_1, \ldots, q_n\} \models \delta(q_0, a)$.*

Definition 3 *We say that a run is accepting if every leaf of the form $(q, 0)$ satisfies that $q \in Q_f$. Notice that a run may have leaves of the form (q, i) with $i > 0$ if $\varnothing \models \delta(q_0, a)$. Those leaves are considered as 'success' leaves in this semantic. The language of a AWA is the set of words which have an accepting run.*

1. Show how to reduce the emptiness problem for an AWA on a one letter alphabet $\{a\}$ whith formalas that are in positive disjunctive normal form to the emptiness problem of a tree automaton .
2. Show how to reduce the emptiness problem for a tree automaton to the emptiness problem of an AWA on a one letter alphabet $\{a\}$. Conclude on the complexity of the emptiness problem for an AWA on a one letter alphabet.

Exercise 2 : Membership

1. Recall the complexity of the uniform membership problem for DFTAs, NFTAs and NFHAs.
2. Prove that **(AlternatingUMembership)** :
   Instance : an AWA $\mathcal{A}$ and a word w
   Question : $w \in L(\mathcal{A})$?
   is in PTIME.

3. Prove that **(HarderUMembership)** :
   Instance : an NFHA $\mathcal{A}$ where the horizontal languages are given by AWA (and not finite automata) and a word w
   Question : $w \in L(\mathcal{A})$?
   is in NP.
4. Let Φ be a propositional formula in CNF with n variables $x_1, ..., x_n$. Construct, in polynomial time, an AWA $\mathcal{A}_\Phi$ whose language is $\{w \in \{0,1\}^n \mid w \models \Phi \text{ i.e. } \Phi_{[x_i \leftarrow w_i]} = \top\}$.

5. Deduce that membership for NFHA where horizontal languages are given by AWA is NP-complete.

Definition 4 (PDL)
The syntax is the following :

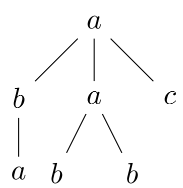
$$\begin{aligned}
\phi &::= a \mid \top \mid \neg\phi \mid \phi \vee \phi \mid \langle\pi\rangle\,\phi && \textit{(position formulae)}\\
\pi &::= \downarrow \mid \rightarrow \mid \pi^{-1} \mid \pi;\pi \mid \pi+\pi \mid \pi^{*} \mid \phi? && \textit{(path formulae)}
\end{aligned}$$

The semantic is defined this way : let t be a tree, we define $[\![\phi]\!]_t$ (resp. $[\![\pi]\!]_t$) as a set of positions of t (resp. a relation on positions of t) by induction on the size of ϕ (resp. π) :

$$\begin{aligned}
[\![a]\!]_t &= \{w \in Pos(t) \mid t(w) = a\} & [\![\downarrow]\!]_t &= \{(w, w.i) \mid w, w.i \in Pos(t)\}\\
[\![\top]\!]_t &= Pos(t) & [\![\rightarrow]\!]_t &= \{(w.i, w.(i+1)) \mid w.i, w.(i+1) \in Pos(t)\}\\
[\![\neg\phi]\!]_t &= Pos(t) \setminus [\![\phi]\!]_t & [\![\pi^{-1}]\!]_t &= [\![\pi]\!]_t^{-1}\\
[\![\phi_1 \vee \phi_2]\!]_t &= [\![\phi_1]\!]_t \cup [\![\phi_2]\!]_t & [\![\pi_1;\pi_2]\!]_t &= [\![\pi_2]\!]_t \circ [\![\pi_1]\!]_t\\
[\![\langle\pi\rangle\,\phi]\!]_t &= [\![\pi]\!]_t^{-1}([\![\phi]\!]_t) & [\![\pi_1+\pi_2]\!]_t &= [\![\pi_1]\!]_t \cup [\![\pi_2]\!]_t\\
[\![\pi^{*}]\!]_t &= [\![\pi]\!]_t^{*} & [\![\phi?]\!]_t &= \Delta_{[\![\phi]\!]_t} = \{(w,w) \mid w \in [\![\phi]\!]_t\}
\end{aligned}$$

Let t be a tree and w, $w' \in Pos(t)$. We note :
— $t, w \models \phi$ *if* $w \in [\![\phi]\!]_t$
— $t \models \phi$ *if* $t, \epsilon \models \phi$ *and we say that t satisfies ϕ*
— $t, w, w' \models \pi$ *if* $(w, w') \in [\![\pi]\!]_t$

Exercise 3 : Warm up
Let t be the tree :



Which formulae are satisfied by t ?

1. $\phi_1 = \neg a \vee \langle\downarrow\rangle \left(\neg\langle\leftarrow\rangle \top \wedge b \wedge \langle\rightarrow^{*}\rangle\,(c \wedge \neg\langle\rightarrow\rangle \top)\right)$
2. $\phi_2 = \neg a \vee \langle\downarrow\rangle \left(\neg\langle\leftarrow\rangle \top \wedge b \wedge \langle(\rightarrow;c?)^{*}\rangle\,(\neg\langle\rightarrow\rangle \top)\right)$
3. $\phi_3 = \langle(a?;\downarrow)^{*}\rangle\,(a \wedge \neg\langle\downarrow\rangle \top)$

Exercise 4 : The power of PDL ?
Give a translation of PDL in MSO which preserves models. That is, given a position formula ϕ (resp. a path formula π), construct a MSO formula $\tilde{\phi}$ (resp. $\tilde{\pi}$) whose set of free variable is $\{X_a \mid a \in \mathcal{F}\} \cup \{x\}$ (resp. $\{X_a \mid a \in \mathcal{F}\} \cup \{x, y\}$) such that $t, w \models \phi$ iff $(P_a(t))_{a\in\mathcal{F}}, w \models \tilde{\phi}$ (resp. $t, w, w' \models \pi$ iff $(P_a(t))_{a\in\mathcal{F}}, w, w' \models \tilde{\pi}$) where $P_a(t) = \{w \in Pos(t) \mid t(w) = a\}$.

Exercise 5 : The exerice we won't have time for
Fix an alphabet $\mathcal{F}$. Give a PDL formula π such that :

- for all tree t and all position p of t, there exists exactly one position q of t such that $(p, q) \in [\![\pi]\!]_t$ (π defines a function on positions).
- for all tree t and position p of t, $(p, q) \in [\![\pi^{*}]\!]_t$ iff q is a position of t such that $t(q) = t(p)$.